COMP 2804 — Solutions Assignment 4

Question 1: On the first page of your assignment, write your name and student number.

Solution:

- Name: Lionel Messi
- Student number: 10

Question 2: Both Alexa and Shelly have an infinite bitstring. Alexa's bitstring is denoted by $a_1a_2a_3\ldots$, whereas Shelly's bitstring is denoted by $s_1s_2s_3\ldots$. Alexa can see her bitstring, but she cannot see Shelly's bitstring. Similarly, Shelly can see her bitstring, but she cannot see Alexa's bitstring. The bits in both bitstrings are uniformly random and independent.

The ladies play the following game: Alexa chooses a positive integer k and Shelly chooses a positive integer ℓ. The game is a *success* if $s_k = 1$ and $a_\ell = 1$. In words, the game is a success if Alexa chooses a position in Shelly's bitstring that contains a 1, and Shelly chooses a position in Alexa's bitstring that contains a 1.

- Assume Alexa chooses $k = 4$ and Shelly chooses $\ell = 7$. Determine the probability that the game is a success.
- Assume Alexa chooses the position, say k, of the leftmost 1 in her bitstring, and Shelly chooses the position, say ℓ, of the leftmost 1 in her bitstring.
  - If $k \neq \ell$, is the game a success?
  - Determine the probability that the game is a success.

Solution: For the first part, Alexa chooses $k = 4$ and Shelly chooses $\ell = 7$. The game is a success if and only if $s_4 = 1$ and $a_7 = 1$. Since the bits are uniformly random and independent, we have

$$\begin{aligned}
\Pr(\text{success}) &= \Pr(s_4 = 1 \text{ and } a_7 = 1) \\
&= \Pr(s_4 = 1) \cdot \Pr(a_7 = 1) \\
&= 1/2 \cdot 1/2 \\
&= 1/4.
\end{aligned}$$

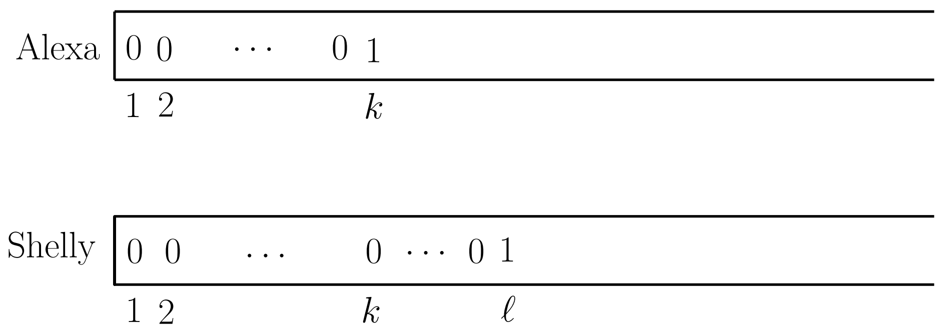
I hope you see that there is nothing magic about the numbers 4 and 7. This part of the question shows that if Alexa chooses any value for k, without looking at her bitstring, and Shelly chooses any value for ℓ, without looking at her bitstring, then the success probability is 1/4.

For the second part, the ladies choose k and ℓ in a more clever way: They choose these values while looking at their bitstrings:

1. Alexa looks at her bitstring and takes for k the first position in which this string has a 1.

2. Shelly looks at her bitstring and takes for ℓ the first position in which this string has a 1.

Assume that $k \neq \ell$. We assume that $k < \ell$. (The case when $k > \ell$ is symmetric.) Look at the following figure:



Since $k < \ell$, Shelly's string has a 0 at position k, i.e., $s_k \neq 1$. Therefore, the game is not a success.

From this, it should be clear that the game is a success if and only if $k = \ell$. Note that the possible value for k are $1, 2, 3, \ldots$. This implies the following:

$$\begin{aligned}
\Pr(\text{success}) &= \sum_{k=1}^{\infty} \Pr\left(s_1 \ldots s_k = \underbrace{0 \ldots 0}_{k-1} 1 \text{ and } a_1 \ldots a_k = \underbrace{0 \ldots 0}_{k-1} 1 \right) \\
&= \sum_{k=1}^{\infty} (1/2)^{2k} \\
&= \sum_{k=1}^{\infty} (1/4)^{k} \\
&= 1/4 + (1/4)^2 + (1/4)^3 + (1/4)^4 + \ldots \\
&= 1/4 \cdot \left(1 + 1/4 + (1/4)^2 + (1/4)^3 + \ldots\right) \\
&= 1/4 \cdot \frac{1}{1 - 1/4} \\
&= 1/3.
\end{aligned}$$

Question 3: Let $n \geq 2$ be an integer. You have n cider bottles $C_1, C_2, \ldots, C_n$ and two beer bottles B_1 and B_2. Consider a uniformly random permutation of these $n + 2$ bottles. The positions in this permutation are numbered $1, 2, \ldots, n + 2$. Define the following two random variables:

$$\begin{aligned}
X &= \text{the position of the first cider bottle,} \\
Y &= \text{the position of the first bottle having index 1.}
\end{aligned}$$

For example, if $n = 5$ and the permutation is

$$B_2, C_5, C_2, C_4, B_1, C_3, C_1,$$

then $X = 2$ and $Y = 5$.

- Determine the expected value $\mathbb{E}(X)$ of the random variable X.
- Determine the expected value $\mathbb{E}(Y)$ of the random variable Y.
  Hint: $\sum_{k=1}^{n+1} k = (n+1)(n+2)/2$ and $\sum_{k=1}^{n+1} k^2 = (n+1)(n+2)(2n+3)/6$.
- Are X and Y independent random variables? Justify your answer.

Solution: We start with the expected value of X. Since there are only two beer bottles, the possible value for X are 1, 2, and 3.

1. The number of permutations of the $n+2$ bottles is $(n+2)!$.
2. How many permutations satisfy $X = 1$? For this, we use the Product Rule:
   (a) Choose a cider bottle and place it at position 1. There are n ways to do this.
   (b) Place the remaining $n+1$ bottles in an arbitrary order at the positions $2, 3, \ldots, n+2$. There are $(n+1)!$ ways to do this.

   Thus, the number of permutations with $X = 1$ is equal to $n \cdot (n+1)!$. It follows that

   $$\Pr(X = 1) = \frac{n \cdot (n+1)!}{(n+2)!} = \frac{n}{n+2}.$$

3. How many permutations satisfy $X = 2$? For this, we use the Product Rule:
   (a) Choose a beer bottle and place it at position 1. There are 2 ways to do this.
   (b) Choose a cider bottle and place it at position 2. There are n ways to do this.
   (c) Place the remaining n bottles in an arbitrary order at the positions $3, 4, \ldots, n+2$. There are $n!$ ways to do this.

   Thus, the number of permutations with $X = 2$ is equal to $2 \cdot n \cdot n!$. It follows that

   $$\Pr(X = 2) = \frac{2 \cdot n \cdot n!}{(n+2)!} = \frac{2n}{(n+1)(n+2)}.$$

4. We can obtain $\Pr(X = 3)$ from

   $$\Pr(X = 3) = 1 - \Pr(X = 1) - \Pr(X = 2).$$

   Alternatively, we can use the Product Rule to determine the number of permutations satisfying $X = 3$:

(a) Place the two beer bottles at positions 1 and 2. There are $2! = 2$ ways to do this.

(b) Place the remaining n bottles in an arbitrary order at the positions $3, 4, \ldots, n+2$. There are $n!$ ways to do this.

Thus, the number of permutations with $X = 3$ is equal to $2 \cdot n!$. It follows that

$$\Pr(X = 3) = \frac{2 \cdot n!}{(n+2)!} = \frac{2}{(n+1)(n+2)}.$$

From this, we get

$$\begin{aligned}
\mathbb{E}(X) &= \sum_{k=1}^{3} k \cdot \Pr(X = k) \\
&= 1 \cdot \Pr(X = 1) + 2 \cdot \Pr(X = 2) + 3 \cdot \Pr(X = 3) \\
&= \frac{n}{n+2} + 2 \cdot \frac{2n}{(n+1)(n+2)} + 3 \cdot \frac{2}{(n+1)(n+2)} \\
&= \frac{n(n+1) + 4n + 6}{(n+1)(n+2)} \\
&= \frac{n^2 + 5n + 6}{(n+1)(n+2)} \\
&= \frac{(n+2)(n+3)}{(n+1)(n+2)} \\
&= \frac{n+3}{n+1} \\
&= 1 + \frac{2}{n+1}.
\end{aligned}$$

Next, we determine the expected value of Y. The possible value for Y are $1, 2, \ldots, n+1$. (Note that Y cannot be equal to $n+2$.)

How many permutations satisfy $Y = k$? For this, we use the Product Rule:

1. Choose one of C_1 and B_1 and place it at position k. There are 2 ways to do this.

2. Place the other of C_1 and B_1 at one of the last $n+2-k$ positions. There are $n+2-k$ ways to do this.

3. Place the remaining n bottles in an arbitrary order at the n available positions. There are $n!$ ways to do this.

Thus, the number of permutations with $Y = k$ is equal to $2 \cdot (n+2-k) \cdot n!$. It follows that

$$\Pr(Y = k) = \frac{2 \cdot (n+2-k) \cdot n!}{(n+2)!} = \frac{2(n+2-k)}{(n+1)(n+2)}.$$

From this, we get

$$
\begin{aligned}
\mathbb{E}(Y) &= \sum_{k=1}^{n+1} k \cdot \Pr(X = k) \\
&= \sum_{k=1}^{n+1} k \cdot \frac{2(n+2-k)}{(n+1)(n+2)} \\
&= \frac{2}{(n+1)(n+2)} \sum_{k=1}^{n+1} k(n+2-k) \\
&= \frac{2}{(n+1)(n+2)} \left((n+2) \sum_{k=1}^{n+1} k - \sum_{k=1}^{n+1} k^2 \right) \\
&= \frac{2}{(n+1)(n+2)} \left((n+2) \cdot \frac{(n+1)(n+2)}{2} - \frac{(n+1)(n+2)(2n+3)}{6} \right) \\
&= (n+2) - \frac{2n+3}{3} \\
&= 1 + n/3.
\end{aligned}
$$

Are X and Y independent? We notice that, if $X = 3$, then $Y \leq 2$. In particular, $X = 3$ and $Y = 3$ cannot simultaneously happen, i.e.,

$$\Pr(X = 3 \text{ and } Y = 3) = 0.$$

From the calculations above, however, neither of $\Pr(X = 3)$ and $\Pr(Y = 3)$ is equal to zero. Thus,

$$\Pr(X = 3 \text{ and } Y = 3) \neq \Pr(X = 3) \cdot \Pr(Y = 3).$$

We conclude that X and Y are not independent.

Question 4: You are given four fair and independent dice, each one having six faces:

1. One die is red and has the numbers $7, 7, 7, 7, 1, 1$ on its faces.
2. One die is blue and has the numbers $5, 5, 5, 5, 5, 5$ on its faces.
3. One die is green and has the numbers $9, 9, 3, 3, 3, 3$ on its faces.
4. One die is yellow and has the numbers $8, 8, 8, 2, 2, 2$ on its faces.

Let c be a color in the set $\{\text{red}, \text{blue}, \text{green}, \text{yellow}\}$. You roll the die of color c. Define the random variable X_c to be the result of this roll.

- For each $c \in \{\text{red}, \text{blue}, \text{green}, \text{yellow}\}$, determine the expected value $\mathbb{E}(X_c)$ of the random variable X_c.

- Let c and c' be two distinct colors in the set $\{\text{red}, \text{blue}, \text{green}, \text{yellow}\}$. Determine

$$\Pr(X_c < X_{c'}) + \Pr(X_c > X_{c'}).$$

- Let c and c' be two distinct colors in the set $\{\text{red}, \text{blue}, \text{green}, \text{yellow}\}$. We say that the die of color c is *better* than the die of color c', if

$$\Pr(X_c > X_{c'}) > 1/2.$$

  For each of the following four questions, justify your answer.

  - Is the red die better than the blue die?
  - Is the blue die better than the green die?
  - Is the green die better than the yellow die?
  - Is the yellow die better than the red die?

  Hint: If you are not surprised by the answers to these four parts of the question, then you made a mistake.

Solution: For each color c, the die has six faces. When you roll this die, each face is the top side with probability $1/6$. If the numbers on the faces of this die are $c_1, \ldots, c_6$, then

$$\begin{aligned} \mathbb{E}(X_c) &= \sum_{i=1}^{6} c_i \cdot 1/6 \\ &= \frac{c_1 + \cdots + c_6}{6}. \end{aligned}$$

Thus, $\mathbb{E}(X_c)$ is just the standard average of the six numbers on this die. It follow that

$$\mathbb{E}(X_{\text{red}}) = \text{average of } 7,7,7,7,1,1 = 5,$$

$$\mathbb{E}(X_{\text{blue}}) = \text{average of } 5,5,5,5,5,5 = 5,$$

$$\mathbb{E}(X_{\text{green}}) = \text{average of } 9,9,3,3,3,3 = 5,$$

$$\mathbb{E}(X_{\text{yellow}}) = \text{average of } 8,8,8,2,2,2 = 5.$$

For the next part, by looking at the dice, you see that none of the numbers occurs on two of the dice. Consider two distinct colors c and c'. Then X_c cannot be equal to $X_{c'}$. Thus, exactly one of $X_c < X_{c'}$ and $X_c > X_{c'}$ must occur. This means that

$$\Pr(X_c < X_{c'}) + \Pr(X_c > X_{c'}) = 1.$$

In order to decide if the red die is better than the blue die, we have to determine $\Pr(X_{\text{red}} > X_{\text{blue}})$.

If we roll the red die once and the blue die once, then there are 36 possible outcomes (these are not all distinct!). How many ways are there so that $X_{\text{red}} > X_{\text{blue}}$:

- X_{red} must be 7 and X_{blue} must be 5. There are $4 \cdot 6 = 24$ ways for this to happen.

Thus, there are 24 outcomes in which $X_{\text{red}} > X_{\text{blue}}$. It follows that

$$\Pr(X_{\text{red}} > X_{\text{blue}}) = \frac{24}{36} = \frac{2}{3} > \frac{1}{2}$$

and we conclude that the red die is better than the blue die.

We do the same for blue and green: In order to decide if the blue die is better than the green die, we have to determine $\Pr(X_{\text{blue}} > X_{\text{green}})$. How many ways are there so that $X_{\text{blue}} > X_{\text{green}}$:

- X_{blue} must be 5 and X_{green} must be 3. There are $6 \cdot 4 = 24$ ways for this to happen.

Thus, there are 24 outcomes in which $X_{\text{blue}} > X_{\text{green}}$. It follows that

$$\Pr(X_{\text{blue}} > X_{\text{green}}) = \frac{24}{36} = \frac{2}{3} > \frac{1}{2}$$

and we conclude that the blue die is better than the green die.

We do the same for green and yellow: In order to decide if the green die is better than the yellow die, we have to determine $\Pr(X_{\text{green}} > X_{\text{yellow}})$. How many ways are there so that $X_{\text{green}} > X_{\text{yellow}}$:

1. X_{green} is 9 and X_{yellow} can have any value. There are $2 \cdot 6 = 12$ ways for this to happen.

2. X_{green} is 3 and X_{yellow} is 2. There are $4 \cdot 3 = 12$ ways for this to happen.

Thus, there are $12 + 12 = 24$ outcomes in which $X_{\text{green}} > X_{\text{yellow}}$. It follows that

$$\Pr(X_{\text{green}} > X_{\text{yellow}}) = \frac{24}{36} = \frac{2}{3} > \frac{1}{2}$$

and we conclude that the green die is better than the yellow die.

We do the same for yellow and red: In order to decide if the yellow die is better than the red die, we have to determine $\Pr(X_{\text{yellow}} > X_{\text{red}})$. How many ways are there so that $X_{\text{yellow}} > X_{\text{red}}$:

1. X_{yellow} is 8 and X_{red} can have any value. There are $3 \cdot 6 = 18$ ways for this to happen.

2. X_{yellow} is 2 and X_{red} is 1. There are $3 \cdot 2 = 6$ ways for this to happen.

Thus, there are $18 + 6 = 24$ outcomes in which $X_{\text{yellow}} > X_{\text{red}}$. It follows that

$$\Pr(X_{\text{yellow}} > X_{\text{red}}) = \frac{24}{36} = \frac{2}{3} > \frac{1}{2}$$

and we conclude that the yellow die is better than the red die.

Conclusion:

red is better than blue,
which is better than green,
which is better than yellow,
which is better than red.

In other words,

the relation "is better than" is not transitive.

These dice are called *non-transitive dice*. There is a wikipedia article about them. How can you use/abuse these dice:

Game 1: You choose one of the colors red, blue, green, and yellow. Then you roll the die of the chosen color. The result of this roll is the amount of money that you win. Which color do you choose?

A first idea is to choose the color for which the expected value is largest. This does not help you, because all expected values are equal. A second idea is to use the definition we made above to define when one color is better than another color. Having this notion of "being better", we choose the "best" color. This does not help either, because there is no "best" color.

Game 2: You choose one of the colors red, blue, green, and yellow. Michiel also chooses one of these four colors. You roll the die of your chosen color once and Michiel rolls the die of his chosen color once. The person with the highest result wins the game.

Since Michiel is very polite, he lets you choose your color first. After Michiel sees which color you have chosen, he chooses his color. Michiel will win this game with probability 2/3.

Question 5: In this question, you are given a fair and independent coin. Let $n \geq 1$ be an integer. Farah flips the coin n times, whereas May flips the coin $n+1$ times. Define the following two random variables:

$$\begin{aligned} X &= \text{the number of heads in Farah's sequence of coin flips,} \\ Y &= \text{the number of heads in May's sequence of coin flips.} \end{aligned}$$

Let A be the event

$$A = \text{"}X < Y\text{"}.$$

- Prove that
$$\Pr(A) = \frac{1}{2^{2n+1}} \sum_{k=0}^{n} \sum_{\ell=k+1}^{n+1} \binom{n}{k} \cdot \binom{n+1}{\ell}.$$
- Define the following two random variables:
$$\begin{aligned} X' &= \text{the number of tails in Farah's sequence of coin flips,} \\ Y' &= \text{the number of tails in May's sequence of coin flips.} \end{aligned}$$
  - What is $X + X'$?

– What is $Y + Y'$?

– Let B be the event

$$B = \text{ `` } X' < Y' \text{ ''}.$$

Explain in plain English and at most two sentences why

$$\Pr(A) = \Pr(B).$$

– Express the event B in terms of the event A.

– Use the results of the previous parts to determine $\Pr(A)$.

- Prove that

$$\sum_{k=0}^{n} \sum_{\ell=k+1}^{n+1} \binom{n}{k} \cdot \binom{n+1}{\ell} = 2^{2n}.$$

Solution: The total number of coin flips is equal to $2n+1$. Therefore, the total number of possible sequences is equal to 2^{2n+1}. How many of these sequences satisfy $X < Y$:

1. Let k be the number of heads in Farah's sequence. The possible values for k are $0, 1, 2, \ldots, n$.

2. Let ℓ be the number of heads in May's sequence. The possible values for ℓ are $k+1, k+2, \ldots, n+1$.

3. For fixed k and ℓ, the number of sequences with $X = k$ and $Y = \ell$ is equal to $\binom{n}{k} \cdot \binom{n+1}{\ell}$.

We conclude that the total number of sequences that satisfy $X < Y$ is equal to

$$\sum_{k=0}^{n} \sum_{\ell=k+1}^{n+1} \binom{n}{k} \cdot \binom{n+1}{\ell}.$$

From this, we get the expression for $\Pr(A)$.

- Obviously, $X + X' = n$.

- Obviously, $Y + Y' = n + 1$.

- Since the coin is fair, everything is symmetric in "heads" and "tails". In other words, if we interchange the roles of "heads" and "tails", then nothing changes. The event B is obtained from the event A by interchanging the roles of "heads" and "tails". Because of this, $\Pr(A) = \Pr(B)$.

- We have the following chain of equivalencies:

$$\begin{aligned} B &\iff X' < Y' \\ &\iff n - X < n + 1 - Y \\ &\iff X > Y - 1 \\ &\iff X \geq Y \qquad \text{(because } X \text{ and } Y \text{ have integer values)} \\ &\iff \overline{A} \end{aligned}$$

- Let $p = \Pr(A)$. By combining the previous parts, we get

$$\begin{aligned} p &= \Pr(A) \\ &= \Pr(B) \\ &= \Pr\left(\overline{A}\right) \\ &= 1 - \Pr(A) \\ &= 1 - p. \end{aligned}$$

If we solve this equation for p, we see that

$$p = \Pr(A) = 1/2.$$

For the last part of the question, we have obtained two expressions for $\Pr(A)$: One expression is the double summation, the other expression is $1/2$. These two expressions must be equal. If you multiply both expressions by 2^{2n+1}, we conclude that

$$\sum_{k=0}^{n} \sum_{\ell=k+1}^{n+1} \binom{n}{k} \cdot \binom{n+1}{\ell} = 2^{2n}.$$

Question 6: Let $n \geq 2$ be an integer and let $a_1, a_2, \ldots, a_n$ be a permutation of the set $\{1, 2, \ldots, n\}$. Define $a_0 = 0$ and $a_{n+1} = 0$, and consider the sequence

$$a_0, a_1, a_2, a_3, \ldots, a_n, a_{n+1}.$$

A position i with $1 \leq i \leq n$ is called *awesome*, if $a_i > a_{i-1}$ and $a_i > a_{i+1}$. In words, i is awesome if the value at position i is larger than both its neighboring values.

For example, if $n = 6$ and the permutation is $2, 5, 4, 3, 1, 6$, we get the sequence

value	0	2	5	4	3	1	6	0
position	0	1	2	3	4	5	6	7

In this case, the positions 2 and 6 are awesome, whereas the positions 1, 3, 4, and 5 are not awesome.

Consider a uniformly random permutation of the set $\{1, 2, \ldots, n\}$ and define the random variable X to be the number of awesome positions. Determine the expected value $\mathbb{E}(X)$ of the random variable X.

Hint: Use indicator random variables.

Solution: We will use indicator random variables $X_1, X_2, \ldots, X_n$, where

$$X_i = \begin{cases} 1 & \text{if position } i \text{ is awesome,} \\ 0 & \text{otherwise.} \end{cases}$$

Since $X = \sum_{i=1}^{n} X_i$, we get

$$\begin{aligned} \mathbb{E}(X) &= \mathbb{E}\left(\sum_{i=1}^{n} X_i\right) \\ &= \sum_{i=1}^{n} \mathbb{E}(X_i). \end{aligned}$$

Since each X_i has value 0 or 1, we have

$$\begin{aligned} \mathbb{E}(X_i) &= \Pr(X_i = 1) \\ &= \Pr(i \text{ is awesome}). \end{aligned}$$

Consider a position i with $2 \leq i \leq n-1$. The position i is awesome if and only if the value at position i is larger than both its neighbors.

Here is an informal argument: Consider the three values at positions $i-1$, i, and $i+1$. There are 6! possible permutations of these. In exactly two of them, position i is awesome. Therefore,

$$\mathbb{E}(X_i) = 2/6 = 1/3.$$

Let us do this more formally: There are $n!$ possible permutations. For how many of these is position i awesome? For this, we will use the Product Rule:

- Choose 3 elements, out of n. There are $\binom{n}{3}$ ways to do this.
- From the 3 chosen elements, place the largest at position i, and place the other 2 at positions $i-1$ and $i+1$. There are 2 ways to do this.
- Place the remaining $n-3$ elements in an arbitrary order in the remaining $n-3$ positions. There are $(n-3)!$ ways to do this.

From this, we see that the number of permutations in which position i is awesome is equal to

$$\binom{n}{3} \cdot 2 \cdot (n-3)! = n!/3.$$

We conclude that

$$\mathbb{E}(X_i) = \frac{n!/3}{n!} = 1/3.$$

Consider the position $i = 1$. This position is awesome if and only if the value at position 1 is larger than its neighbor at position 2. If you understood the reasoning above, then you won't have any difficulty to convince yourself that the number of permutations in which position 1 is awesome is equal to

$$\binom{n}{2} \cdot (n-2)! = n!/2.$$

We conclude that

$$\mathbb{E}(X_1) = \frac{n!/2}{n!} = 1/2.$$

Using the same reasoning, we get

$$\mathbb{E}(X_n) = 1/2.$$

If we put everything together, we get

$$\begin{aligned}
\mathbb{E}(X) &= \sum_{i=1}^{n} \mathbb{E}(X_i) \\
&= 1/2 + (n-2) \cdot 1/3 + 1/2 \\
&= (n+1)/3.
\end{aligned}$$

Question 7: If X is a random variable that can take any value in $\{1, 2, 3, \ldots\}$, and if A is an event, then the *conditional expected value* $\mathbb{E}(X \mid A)$ is defined as

$$\mathbb{E}(X \mid A) = \sum_{k=1}^{\infty} k \cdot \Pr(X = k \mid A).$$

In words, $\mathbb{E}(X \mid A)$ is the expected value of X, when you are given that the event A occurs.

You roll a fair die repeatedly, and independently, until you see the number 6. Define the random variable X to be the number of times you roll the die (this includes the last roll, in which you see the number 6). We have seen in class that $\mathbb{E}(X) = 6$. Let A be the event

$$A = \text{"the results of all rolls are even numbers"}.$$

Determine the conditional expected value $\mathbb{E}(X \mid A)$.
Hint: The answer is not what you expect. We have seen in class that $\sum_{k=1}^{\infty} k \cdot x^{k-1} = 1/(1-x)^2$.

Solution: In class, we have seen the following: Consider an experiment that is successful with probability p. We repeat the experiment, independently, until it is successful for the first time. The expected number of times we do the experiment is equal to $1/p$.

In this question, the experiment is rolling a die and we are successful if we roll a 6. We denote the number of rolls by X. The success probability is $1/6$ and, therefore, $\mathbb{E}(X) = 6$.

The event A says that we always roll an even number. This means that, on every roll, we have one of 2, 4, and 6, and we stop at the first 6. Thus, the success probability becomes 1/3, suggesting that $\mathbb{E}(X \mid A)$ is equal to 3. Most people (including Michiel) think that this must be the correct answer. Below, we will see that this is not the case. The conclusion is that probability theory is very strange.

The sample space is the set of all sequences of rolls that can occur:

$$S = \{r_1 r_2 \ldots r_k : k \geq 1, r_1 \neq 6, \ldots, r_{k-1} \neq 6, r_k = 6\}.$$

The event A is the subset of S in which all rolls are even:

$$A = \{r_1 r_2 \ldots r_k : k \geq 1, r_1 \in \{2,4\}, \ldots, r_{k-1} \in \{2,4\}, r_k = 6\}.$$

We will need $\Pr(A)$, so we start by determining this probability:

$$\begin{aligned}
\Pr(A) &= \sum_{k=1}^{\infty} \Pr\left(\underbrace{2 \text{ or } 4, \ldots, 2 \text{ or } 4}_{k-1}, 6\right) \\
&= \sum_{k=1}^{\infty} (1/3)^{k-1} \cdot 1/6 \\
&= 1/6 \cdot \left(1 + 1/3 + (1/3)^2 + (1/3)^3 + (1/3)^4 + \cdots\right) \\
&= 1/6 \cdot \frac{1}{1 - 1/3} \\
&= 1/4.
\end{aligned}$$

To determine $\mathbb{E}(X \mid A)$, we need $\Pr(X = k \mid A)$:

$$\begin{aligned}
\Pr(X = k \mid A) &= \frac{\Pr(X = k \wedge A)}{\Pr(A)} \\
&= \frac{\Pr(X = k \wedge A)}{1/4} \\
&= 4 \cdot \Pr(X = k \wedge A) \\
&= 4 \cdot \Pr\left(\underbrace{2 \text{ or } 4, \ldots, 2 \text{ or } 4}_{k-1}, 6\right) \\
&= 4 \cdot (1/3)^{k-1} \cdot 1/6 \\
&= 2/3 \cdot (1/3)^{k-1}.
\end{aligned}$$

The rest is now obtained by doing the algebra and using the hint:

$$\mathbb{E}(X \mid A) = \sum_{k=1}^{\infty} k \cdot \Pr(X = k \mid A)$$

$$
\begin{aligned}
&= \sum_{k=1}^{\infty} k \cdot 2/3 \cdot (1/3)^{k-1} \\
&= 2/3 \sum_{k=1}^{\infty} k \cdot (1/3)^{k-1} \\
&= 2/3 \cdot \left(\frac{1}{1-1/3} \right)^2 \\
&= 3/2.
\end{aligned}
$$

Here is an alternative solution. Assume we roll a fair die repeatedly and independently, until we see one of the numbers 1, 3, 5, 6. Let Y denote the number of rolls (this includes the last roll, in which we see one of 1, 3, 5, 6). The success probability in one roll is equal to $p = 4/6 = 2/3$. Therefore, we know from class that

$$\mathbb{E}(Y) = 3/2.$$

For each $i \in \{1, 3, 5, 6\}$, define the event

$$B_i = \text{ "the result of the last roll is } i\text{"}.$$

By symmetry, we have

$$\Pr(B_1) = \Pr(B_3) = \Pr(B_5) = \Pr(B_6) = 1/4$$

and

$$\mathbb{E}(Y \mid B_1) = \mathbb{E}(Y \mid B_3) = \mathbb{E}(Y \mid B_5) = \mathbb{E}(Y \mid B_6).$$

Next, we observe that

$$\mathbb{E}(X \mid A) = \mathbb{E}(Y \mid B_6).$$

The event "$Y = k$" happens if and only if one of the four events "$Y = k \wedge B_i$", for $i \in \{1, 3, 5, 6\}$, happens. Since the latter four events are pairwise disjoint, we have (the sum is over $i = 1, 3, 5, 6$)

$$
\begin{aligned}
\Pr(Y = k) &= \sum_i \Pr(Y = k \wedge B_i) \\
&= \sum_i \Pr(Y = k | B_i) \cdot \Pr(B_i).
\end{aligned}
$$

By combining everything, we get

$$
\begin{aligned}
3/2 &= \mathbb{E}(Y) \\
&= \sum_{k=1}^{\infty} k \cdot \Pr(Y = k)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=1}^{\infty} k \sum_{i} \Pr\left(Y = k | B_i\right) \cdot \Pr\left(B_i\right) \\
&= \sum_{k=1}^{\infty} k \sum_{i} \Pr\left(Y = k | B_i\right) \cdot 1/4 \\
&= 1/4 \cdot \sum_{i} \sum_{k=1}^{\infty} k \cdot \Pr\left(Y = k | B_i\right) \\
&= 1/4 \cdot \sum_{i} \mathbb{E}\left(Y | B_i\right) \\
&= 1/4 \cdot \sum_{i} \mathbb{E}(X | A) \\
&= 1/4 \cdot 4 \cdot \mathbb{E}\left(X | A\right) \\
&= \mathbb{E}\left(X | A\right).
\end{aligned}
$$