COMP 2804 — Assignment 1

Due: Thursday September 27, before 11:55pm.

Assignment Policy:

- Your assignment must be submitted as one single PDF file through cuLearn.
- **Late assignments will not be accepted. I will not reply to emails of the type "my internet connection broke down at 11:53pm" or "my scanner stopped working at 11:54pm".**
- You are encouraged to collaborate on assignments, but at the level of discussion only. When writing your solutions, you must do so in your own words.
- Past experience has shown conclusively that those who do not put adequate effort into the assignments do not learn the material and have a probability near 1 of doing poorly on the exams.
- When writing your solutions, you must follow the guidelines below.
  - You must justify your answers.
  - The answers should be concise, clear and neat.
  - When presenting proofs, every step should be justified.

Question 1:

- Write your name and student number.

Question 2: Let $n \geq 2$ be an integer.

- Determine the number of strings consisting of n characters, where each character is an element of the set $\{a, b, 0\}$.
- Let S be a set consisting of n elements. Determine the number of ordered pairs (A, B), where $A \subseteq S$, $B \subseteq S$, and $A \cap B = \emptyset$.
- Let S be a set consisting of n elements. Consider ordered pairs (A, B), where $A \subseteq S$, $B \subseteq S$, and $|A \cap B| = 1$. Prove that the number of such pairs is equal to $n \cdot 3^{n-1}$.

Question 3: Consider 10 male students $M_1, M_2, \ldots, M_{10}$ and 7 female students $F_1, F_2, \ldots, F_7$. Assume these 17 students are arranged on a horizontal line such that no two female students are standing next to each other. How many such arrangements are there? (The order of the students matters.)
Hint: Use the Product Rule. What is easier to count: Placing the female students first and then the male students, or placing the male students first and then the female students?

Question 4: Elisa Kazan[1] has a set $\{C_1, C_2, \ldots, C_{50}\}$ consisting of 50 cider bottles. She divides these bottles among 5 friends, so that each friend receives a subset consisting of 10 bottles. Determine the number of ways in which Elisa can divide the bottles.

Question 5: Let $f \geq 2$, $m \geq 2$, and $k \geq 2$ be integers such that $k \leq f$ and $k \leq m$. The Carleton Computer Science program has f female students and m male students. The Carleton Computer Science Society has a Board of Directors consisting of k students. At least one of the board members is female and at least one of the board members is male. Determine the number of ways in which a Board of Directors can be chosen.

Question 6: You have won the first prize in the *Louis van Gaal Impersonation Contest*[2]. When you arrive at Louis' home to collect your prize, you see n beer bottles $B_1, B_2, \ldots, B_n$, n cider bottles $C_1, C_2, \ldots, C_n$, and n wine bottles $W_1, W_2, \ldots, W_n$. Here, n is an integer with $n \geq 2$. Louis tells you that your prize consists of one beer bottle of your choice, one cider bottle of your choice, and one wine bottle of your choice.

Prove that

$$n^3 = (n-1)^3 + 3(n-1)^2 + 3(n-1) + 1,$$

by counting, in two different ways, the number of ways in which you can choose your prize.

Question 7: Let $a \geq 0$, $b \geq 0$, and $n \geq 0$ be integers, and consider the set $S = \{1, 2, 3, \ldots, a+b+n+1\}$.

- How many subsets of size $a+b+1$ does S have?
- Let k be an integer with $0 \leq k \leq n$. Consider subsets T of S such that $|T| = a+b+1$ and the $(a+1)$-st smallest element in T is equal to $a+k+1$. How many such subsets T are there?
- Use the above results to prove that

$$\sum_{k=0}^{n} \binom{a+k}{k} \binom{b+n-k}{n-k} = \binom{a+b+n+1}{n}.$$

[1] President of the Carleton Computer Science Society

[2] Louis van Gaal has been coach of AZ, Ajax, Barcelona, Bayern München, Manchester United, and the Netherlands.

Question 8: In this exercise, we consider strings that can be obtained by reordering the letters of the word ENGINE.

- Determine the number of strings that can be obtained.
- Determine the number of strings in which the two letters E are next to each other.
- Determine the number of strings in which the two letters E are not next to each other and the two letters N are not next to each other.

(You do not get marks if you write out all possible strings. You must use the counting rules that you learned in class.)

Question 9: The square in the left figure below is divided into nine cells. In each cell, we write one of the numbers -1, 0, and 1.

Use the Pigeonhole Principle to prove that, among the rows, columns, and main diagonals, there exist two that have the same sum. For example, in the right figure below, both main diagonals have sum 0. (Also, the two topmost rows both have sum 1, whereas the bottom row and the right column both have sum -2.)

0	1	0
1	1	-1
-1	0	-1

Question 10: Let $d \geq 1$ be an integer. A point p in $\mathbb{R}^d$ is represented by its d real coordinates as $p = (p_1, p_2, \ldots, p_d)$. The *midpoint* of two points $p = (p_1, p_2, \ldots, p_d)$ and $q = (q_1, q_2, \ldots, q_d)$ is the point

$$\left(\frac{p_1 + q_1}{2}, \frac{p_2 + q_2}{2}, \ldots, \frac{p_d + q_d}{2} \right).$$

Let P be a set of $2^d + 1$ points in $\mathbb{R}^d$, all of which have integer coordinates.

Use the Pigeonhole Principle to prove that this set P contains two distinct elements whose midpoint has integer coordinates.

Hint: The sum of two even integers is even, and the sum of two odd integers is even.