COMP 2804 — Assignment 3

Due: Thursday November 15, before 11:55pm, through cuLearn.

Assignment Policy:

- Your assignment must be submitted as one single PDF file through cuLearn.
- **Late assignments will not be accepted. I will not reply to emails of the type "my internet connection broke down at 11:53pm" or "my scanner stopped working at 11:54pm".**
- You are encouraged to collaborate on assignments, but at the level of discussion only. When writing your solutions, you must do so in your own words.
- Past experience has shown conclusively that those who do not put adequate effort into the assignments do not learn the material and have a probability near 1 of doing poorly on the exams.
- When writing your solutions, you must follow the guidelines below.
  - You must justify your answers.
  - The answers should be concise, clear and neat.
  - When presenting proofs, every step should be justified.

Question 1: Write your name and student number.

Question 2: You are given a box that contains the 8 lowercase letters a, b, c, d, e, f, g, h and the 5 uppercase letters V, W, X, Y, Z.

- You choose 4 letters from the box: These letters are chosen in 4 steps, and in each step, you choose a uniformly random letter from the box; this letter is removed from the box.
  - What is the sample space?
  - Define the event

    $$A = \text{"the 4-th letter chosen is an uppercase letter "}.$$

    Determine $\Pr(A)$.
- You choose 4 letters from the box: These letters are chosen in 4 steps, and in each step, you choose a uniformly random letter from the box; this letter is *not* removed from the box.
  - What is the sample space?

– Define the event

$$B = \text{“the 4-th letter chosen is an uppercase letter”}.$$

Determine $\Pr(B)$.

Question 3: Donald Trump wants to hire two secretaries. There are n applicants $a_1, a_2, \dots, a_n$ for these jobs, where $n \geq 2$ is an integer. Each of these applicants has a uniformly random birthday, and all birthdays are independent. (We ignore leap years.)

Since Donald is too busy making America great again, he does not have time to interview the applicants. Instead, he uses the following strategy: If there is an index i such that a_i and a_{i+1} have the same birthday, then he chooses the smallest such index i and hires a_i and a_{i+1}. In this case, the hiring process is a *success*. If such an index i does not exist, then nobody is hired and the hiring process is a *total disaster*.

- Determine the probability that the hiring process is a success.

Question 4: You roll a fair die once. Define the events

$$\begin{aligned} A &= \text{“the result is one of the numbers 1, 3, and 4”}, \\ B &= \text{“the result is one of the numbers 3, 4, 5, and 6”}. \end{aligned}$$

Are the events A and B independent?

Question 5: You are given a fair red die and a fair blue die, and roll both dice uniformly at random and independently of each other. Define the events

$$\begin{aligned} A &= \text{“the sum of the results is at least 9”}, \\ B &= \text{“at least one of the two rolls results in a 2”}, \\ C &= \text{“at least one of the two rolls results in a 5”}. \end{aligned}$$

- Determine $\Pr(A)$, $\Pr(B)$, and $\Pr(C)$.
- Determine $\Pr(B \mid C)$.
- Are the events A and B independent?
- Are the events A and C independent?

Question 6: Let A and B be two events in some probability space $(S, \Pr)$ such that $\Pr(A) = 2/5$ and $\Pr\left(\overline{A \cup B}\right) = 3/10$.

- Assume that A and B are disjoint. Determine $\Pr(B)$.

- Assume that A and B are independent. Determine $\Pr(B)$.

Question 7: You are given a fair die. For any integer $n \geq 1$, you roll this die n times (the rolls are independent). Define the events

$$A_n = \text{“the sum of the results of the } n \text{ rolls is even”}$$

and

$$B_n = \text{“the last roll in the sequence of } n \text{ rolls results in an even number”},$$

and their probabilities

$$p_n = \Pr(A_n)$$

and

$$q_n = \Pr(B_n).$$

- Determine p_1.
- For any integer $n \geq 1$, determine q_n.
- For any integer $n \geq 2$, express the event A_n in terms of the events A_{n-1} and B_n.
- Use the previous parts to determine p_n for any integer $n \geq 2$.

Question 8: Let n and k be integers with $1 \leq n \leq k \leq 2n$. In this exercise, you will prove that

$$\sum_{i=k-n}^{n} \binom{k}{i} \binom{2n-k}{n-i} = \binom{2n}{n}. \tag{1}$$

Jim is working on the first assignment for the course COMP 4999 (Computational Aspects of Growing Cannabis). There are $2n$ questions on this assignment and each of them is worth 1 mark. Two minutes before the deadline, Jim has completed the first k questions. Jim is very smart and all answers to these k questions are correct. Jim knows that the instructor, Professor Mary Juana, does not accept late submissions. Because of this, Jim leaves the last $2n-k$ questions blank and hands in his assignment.

Tri is a TA for this course[1]. Since Tri is lazy, he does not want to mark all questions. Instead, he chooses a uniformly random subset of n questions out of the $2n$ questions, and only marks the n chosen questions. For each correct answer, Tri gives 2 marks, whereas he gives 0 marks for each wrong (or blank) answer.

For each integer $i \geq 0$, define the event

$$A_i = \text{“Jim receives exactly } 2i \text{ marks for his assignment”}.$$

[1] Tri is also a TA for COMP 2804

- Determine the value of the summation $\sum_i \Pr(A_i)$. Explain your answer in plain English.
- Determine all values of i for which the event A_i is non-empty. For each such value i, determine $\Pr(A_i)$.
- Prove that (1) holds by combining the results of the previous parts.

Question 9: Let a and z be integers with $a > z \geq 1$, and let p be a real number with $0 < p < 1$. Alexa[2] and Zoltan[3] play a game consisting of several rounds. In each round,

1. Alexa receives a points with probability p and 0 points with probability $1 - p$,
2. Zoltan receives z points (with probability 1).

We assume that the results of different rounds are independent.

- Define the event

  $$A = \text{"in one round, Alexa receives more points than Zoltan"}.$$

  We say that Alexa is a *better player* than Zoltan, if $\Pr(A) > 1/2$.

  For which values of p is Alexa a better player than Zoltan?

- Assume that $a = 3$, $z = 2$, and p is chosen such that $p > 1/2$ and $p^2 < 1/2$. (For example, $p = (\sqrt{5} - 1)/2$.)
  - Is Alexa a better player than Zoltan?
  - Alexa and Zoltan play a game consisting of two rounds. We consider the total number of points that each player wins during these two rounds. Define the event

    $$B = \text{"in two rounds, Alexa receives more points than Zoltan"}.$$

    Prove that $\Pr(B) < 1/2$. (This seems to suggest that Zoltan is a better player than Alexa.)

- Let n be a large integer, and assume that $a = n + 1$, $z = n$, and p is chosen very close to (but less than) 1. (For example, $n = 500$ and $p = 0.99$.)
  - Is Alexa a better player than Zoltan?
  - Alexa and Zoltan play a game consisting of n rounds. We consider the total number of points that each player wins during these n rounds. Define the event

    $$C = \text{"in } n \text{ rounds, Alexa receives more points than Zoltan"}.$$

    Prove that $\Pr(C) = p^n$. (If $n = 500$ and $p = 0.99$, then $p^n \approx 0.0066$. This seems to suggest that Zoltan is a *much* better player than Alexa.)

[2] your friendly TA
[3] another friendly TA