COMP 2804 — Solutions Assignment 3

Question 1: Write your name and student number.

Solution:

- Name: Lionel Messi
- Student number: 10

Question 2: You are given a box that contains the 8 lowercase letters a, b, c, d, e, f, g, h and the 5 uppercase letters V, W, X, Y, Z.

- You choose 4 letters from the box: These letters are chosen in 4 steps, and in each step, you choose a uniformly random letter from the box; this letter is removed from the box.
  - What is the sample space?
  - Define the event

    $$A = \text{"the 4-th letter chosen is an uppercase letter "}.$$

    Determine $\Pr(A)$.
- You choose 4 letters from the box: These letters are chosen in 4 steps, and in each step, you choose a uniformly random letter from the box; this letter is *not* removed from the box.
  - What is the sample space?
  - Define the event

    $$B = \text{"the 4-th letter chosen is an uppercase letter"}.$$

    Determine $\Pr(B)$.

Solution: We will use the set

$$\mathcal{P} = \{a, b, c, d, e, f, g, h, V, W, X, Y, Z\}.$$

Note that $|\mathcal{P}| = 13$.

We start with the first part. The sample space is the set

$$S = \{(x_1, x_2, x_3, x_4) : x_1, x_2, x_3, x_4 \in \mathcal{P} \text{ and } |\{x_1, x_2, x_3, x_4\}| = 4\}.$$

Note that $|\{x_1, x_2, x_3, x_4\}| = 4$ is a fancy way of saying that x_1, x_2, x_3, x_4 are pairwise distinct.

The event A is the set of all elements $(x_1, x_2, x_3, x_4) \in S$ for which x_4 is an uppercase letter of $\mathcal{P}$. Since all choices are uniformly at random, we have

$$\Pr(A) = |A|/|S|.$$

1. What is the size of S: This follows from the Product Rule:

$$|S| = 13 \cdot 12 \cdot 11 \cdot 10 = 17160.$$

2. **Complicated way to determine the size of A:** We consider the following four cases:

   (a) All of x_1, x_2, x_3 are lowercase letters.
   There are 8 choices for x_1, 7 choices for x_2, 6 choices for x_3, and, since x_4 is an uppercase letter, 5 choices for x_4. Thus, the number of possibilities for this is case is

   $$8 \cdot 7 \cdot 6 \cdot 5 = 1680.$$

   (b) Among x_1, x_2, x_3, there are 2 lowercase letters.
   There are $\binom{3}{2} = 3$ choices for the positions of these 2 lowercase letters, 8 choices for one of them, 7 choices for the other, 5 choices for the uppercase letter among x_1, x_2, x_3, and 4 choices for x_4. Thus, the number of possibilities for this is case is

   $$3 \cdot 8 \cdot 7 \cdot 5 \cdot 4 = 3360.$$

   (c) Among x_1, x_2, x_3, there is 1 lowercase letter.
   There are $\binom{3}{1} = 3$ choices for the position of this lowercase letter, 8 choices for the lowercase letter, 5 choices for the first uppercase letter, 4 choices for the second uppercase letter, and 3 choices for x_4. Thus, the number of possibilities for this case is

   $$3 \cdot 8 \cdot 5 \cdot 4 \cdot 3 = 1440.$$

   (d) All of x_1, x_2, x_3 are uppercase letters.
   There are 5 choices for x_1, 4 choices for x_2, 3 choices for x_3, and 2 choices for x_4. Thus, the number of possibilities for this case is

   $$5 \cdot 4 \cdot 3 \cdot 2 = 120.$$

   (e) We conclude that

   $$|A| = 1680 + 3360 + 1440 + 120 = 6600.$$

3. **Easy way to determine the size of A:** There are 5 ways to choose x_4. Given x_4, there are 12 ways to choose x_3. Given x_3, there are 11 ways to choose x_2. Given x_2, there are 10 ways to choose x_1. Therefore,

$$|A| = 5 \cdot 12 \cdot 11 \cdot 10 = 6600.$$

4. We conclude that

$$\Pr(A) = |A|/|S| = 6600/17160 = 5/13 \approx 0.385.$$

For the second part, the sample space is the set

$$S = \{(x_1, x_2, x_3, x_4) : x_1, x_2, x_3, x_4 \in \mathcal{P}\}.$$

The event B is the set of all elements $(x_1, x_2, x_3, x_4) \in S$ for which x_4 is an uppercase letter of $\mathcal{P}$. Since all choices are uniformly at random, we have

$$\Pr(B) = |B|/|S|.$$

1. What is the size of S: This follows from the Product Rule:

$$|S| = 13 \cdot 13 \cdot 13 \cdot 13 = 13^4.$$

2. What is the size of B: This follows from the Product Rule:

$$|B| = 13 \cdot 13 \cdot 13 \cdot 5 = 13^3 \cdot 5.$$

3. We conclude that

$$\Pr(B) = |B|/|S| = 13^3 \cdot 5/13^4 = 5/13 \approx 0.385.$$

A less formal (but also correct) approach is as follows: The event B is completely determined by x_4; we do not care about x_1, x_2, x_3. Therefore, $\Pr(B)$ is the probability that we pick an uppercase letter from the set $\mathcal{P}$, which is $5/|\mathcal{P}| = 5/13$.

Note: For event A, we did sampling without replacement, for event B, we did sampling with replacement. In both cases, we get the same answer.

Question 3: Donald Trump wants to hire two secretaries. There are n applicants $a_1, a_2, \ldots, a_n$ for these jobs, where $n \geq 2$ is an integer. Each of these applicants has a uniformly random birthday, and all birthdays are independent. (We ignore leap years.)

Since Donald is too busy making America great again, he does not have time to interview the applicants. Instead, he uses the following strategy: If there is an index i such that a_i and a_{i+1} have the same birthday, then he chooses the smallest such index i and hires a_i and a_{i+1}. In this case, the hiring process is a *success*. If such an index i does not exist, then nobody is hired and the hiring process is a *total disaster*.

- Determine the probability that the hiring process is a success.

Solution: For each $i = 1, 2, \ldots, n$, let b_i denote a_i's birthday. The sample space is the set

$$S = \{(b_1, b_2, \ldots, b_n) : \text{ each } b_i \in \{1, 2, \ldots, 365\}\}.$$

By the Product Rule, we have

$$|S| = 365^n.$$

We define the event A to be "the hiring process is a success". It is easier to consider the complement, i.e., the event $\overline{A}$ which says that the hiring process is a total disaster. Note that $(b_1, b_2, \ldots, b_n) \in \overline{A}$ if and only if

$$b_1 \neq b_2, b_2 \neq b_3, b_3 \neq b_4, \ldots, b_{n-1} \neq b_n.$$

How many such sequences of birthdays are there: There are 365 choices for b_1, and 364 choices for each of the other $n-1$ birthdays (because each one must be different from the previous birthday). Therefore,

$$|\overline{A}| = 365 \cdot 364^{n-1}.$$

We conclude that

$$\Pr(A) = 1 - \Pr\left(\overline{A}\right) = 1 - |\overline{A}|/|S| = 1 - (364/365)^{n-1}.$$

Question 4: You roll a fair die once. Define the events

$$\begin{aligned} A &= \text{"the result is one of the numbers 1, 3, and 4"}, \\ B &= \text{"the result is one of the numbers 3, 4, 5, and 6"}. \end{aligned}$$

- Are the events A and B independent? Justify your answer using the definition of independence.
- Are the events A and B independent? Justify your answer using the definition of conditional probability.

Solution: The sample space is the set

$$S = \{1, 2, 3, 4, 5, 6\},$$

and the events A and B are the sets

$$A = \{1, 3, 4\}$$

and

$$B = \{3, 4, 5, 6\}.$$

Note that

$$A \cap B = \{3, 4\}.$$

We have

$$\Pr(A) = |A|/|S| = 3/6 = 1/2,$$
$$\Pr(B) = |B|/|S| = 4/6 = 2/3,$$

and

$$\Pr(A \cap B) = |A \cap B|/|S| = 2/6 = 1/3.$$

For the first part, we verify the equation

$$\Pr(A \cap B) \stackrel{?}{=} \Pr(A) \cdot \Pr(B).$$

By plugging in the probabilities that we determined above, this becomes

$$1/3 \stackrel{?}{=} 1/2 \cdot 2/3,$$

which is true. Therefore, the events A and B are independent.

For the second part, we verify the equation

$$\Pr(A|B) \stackrel{?}{=} \Pr(A),$$

which is the same as

$$\Pr(A \cap B)/\Pr(B) \stackrel{?}{=} \Pr(A).$$

By plugging in the probabilities that we determined above, this becomes

$$(1/3)/(2/3) \stackrel{?}{=} 1/2,$$

which is true. Therefore, the events A and B are independent.

Note that we could also have verified the equation

$$\Pr(B|A) \stackrel{?}{=} \Pr(B).$$

Question 5: You are given a fair red die and a fair blue die, and roll both dice uniformly at random and independently of each other. Define the events

$$\begin{aligned} A &= \text{"the sum of the results is at least 9"}, \\ B &= \text{"at least one of the two rolls results in a 2"}, \\ C &= \text{"at least one of the two rolls results in a 5"}. \end{aligned}$$

- Determine $\Pr(A)$, $\Pr(B)$, and $\Pr(C)$.
- Determine $\Pr(B \mid C)$.
- Are the events A and B independent?
- Are the events A and C independent?

Solution: In the matrix below, the leftmost column indicates the result of the red die, the top row indicates the result of the blue die, and each entry is the sum of the results of the two corresponding rolls.

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

- The sample space has size 36.
  - From the table, we see that $|A| = 10$. Therefore,
  $$\Pr(A) = |A|/|S| = 10/36 = 5/18.$$
  - Since $|B| = 6 + 6 - 1 = 11$, we have
  $$\Pr(B) = |B|/|S| = 11/36.$$
  - Since $|C| = 6 + 6 - 1 = 11$, we have
  $$\Pr(C) = |C|/|S| = 11/36.$$
- Since $B \cap C = \{(2,5),(5,2)\}$, we have
$$\Pr(B \cap C) = |B \cap C|/|S| = 2/36 = 1/18,$$
which implies that
$$\Pr(B \mid C) = \frac{\Pr(B \cap C)}{\Pr(C)} = \frac{2/36}{11/36} = 2/11.$$
- To decide if the events A and B are independent, we verify the equation
$$\Pr(A \cap B) \stackrel{?}{=} \Pr(A) \cdot \Pr(B).$$
Since $A \cap B = \emptyset$, we have $\Pr(A \cap B) = 0$. Since $\Pr(A) \neq 0$ and $\Pr(B) \neq 0$, it follows that
$$\Pr(A \cap B) \neq \Pr(A) \cdot \Pr(B).$$
Therefore, A and B are not independent.
- Since $A \cap C = \{(5,4),(5,5),(5,6),(4,5),(6,5)\}$, we have $\Pr(A \cap C) = 5/36$. Since $\Pr(A) \cdot \Pr(C) = 5/18 \cdot 11/36 \neq 5/36$, the events A and C are not independent.

Question 6: Let A and B be two events in a sample space S such that $\Pr(A) = 2/5$ and $\Pr\left(\overline{A \cup B}\right) = 3/10$.

- Assume that A and B are disjoint. Determine $\Pr(B)$.
- Assume that A and B are independent. Determine $\Pr(B)$.

Solution: We are given that

$$\Pr(A) = 2/5.$$

Since $\Pr\left(\overline{A \cup B}\right) = 3/10$, we have

$$\Pr(A \cup B) = 7/10.$$

We have seen in class that

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B). \tag{1}$$

- We assume that A and B are disjoint, i.e., $A \cap B = \emptyset$. Then $\Pr(A \cap B) = 0$, and (1) becomes

  $$7/10 = 2/5 + \Pr(B) - 0,$$

  i.e., $\Pr(B) = 3/10$.

- Now we assume that A and B are independent, i.e.,

  $$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B).$$

  Then (1) becomes

  $$\frac{7}{10} = \frac{2}{5} + \Pr(B) - \frac{2}{5} \cdot \Pr(B).$$

  After some algebra, we see that $\Pr(B) = 1/2$.

Question 7: You are given a fair die. For any integer $n \geq 1$, you roll this die n times (the rolls are independent). Define the events

$$A_n = \text{"the sum of the results of the } n \text{ rolls is even"}$$

and

$$B_n = \text{"the last roll in the sequence of } n \text{ rolls results in an even number"},$$

and their probabilities

$$p_n = \Pr\left(A_n\right)$$

and

$$q_n = \Pr\left(B_n\right).$$

- Determine p_1.
- For any integer $n \geq 1$, determine q_n.

- For any integer $n \geq 2$, express the event A_n in terms of the events A_{n-1} and B_n.
- Use the previous parts to determine p_n for any integer $n \geq 2$.

Solution:

- To determine p_1: We roll the die once, resulting in a uniformly random element from the set $\{1, 2, 3, 4, 5, 6\}$. Since 3 of these 6 elements are even, $p_1 = 1/2$.
- To determine q_n: If we roll a die n times, then there are 6^n many possible outcomes. The number of outcomes in which the last roll is even is equal to $6^{n-1} \cdot 3$. Therefore,
$$q_n = \frac{6^{n-1} \cdot 3}{6^n} = \frac{1}{2}.$$
Note that this answer makes perfect sense: We roll a die n times and only care about the last roll (in other words, we can ignore the first $n-1$ rolls).
- How to express A_n in terms of A_{n-1} and B_n: A_n says that the sum of n rolls is even. There are two possibilities for this to happen:
  - The sum of the first $n-1$ rolls is even and the last roll is even as well.
  - The sum of the first $n-1$ rolls is odd and the last roll is odd as well.

  This implies that
$$A_n = (A_{n-1} \wedge B_n) \vee \left(\overline{A}_{n-1} \wedge \overline{B}_n\right).$$
- To determine p_n for any integer $n \geq 2$: From the previous part, we get
$$p_n = \Pr(A_n) = \Pr\left((A_{n-1} \wedge B_n) \vee \left(\overline{A}_{n-1} \wedge \overline{B}_n\right)\right).$$
Since the two events $A_{n-1} \wedge B_n$ and $\overline{A}_{n-1} \wedge \overline{B}_n$ are disjoint, we get
$$p_n = \Pr(A_{n-1} \wedge B_n) + \Pr\left(\overline{A}_{n-1} \wedge \overline{B}_n\right).$$
The event A_{n-1} only cares about the first $n-1$ rolls, whereas the event B_n only cares about the n-th roll. Therefore, these events are independent. This gives
$$\Pr(A_{n-1} \wedge B_n) = \Pr(A_{n-1}) \cdot \Pr(B_n) = p_{n-1} \cdot q_n = p_{n-1} \cdot \frac{1}{2}$$
and
$$\Pr\left(\overline{A}_{n-1} \wedge \overline{B}_n\right) = \Pr\left(\overline{A}_{n-1}\right) \cdot \Pr\left(\overline{B}_n\right) = (1 - p_{n-1}) \cdot (1 - q_n) = (1 - p_{n-1}) \cdot \frac{1}{2}.$$
We conclude that
$$p_n = p_{n-1} \cdot \frac{1}{2} + (1 - p_{n-1}) \cdot \frac{1}{2} = \frac{1}{2}.$$
Note: Something strange happened: We used a recurrence for the events A_n, from which we derived a "recurrence" for p_n. Since the terms p_{n-1} cancel, we did not obtain a "real" recurrence.

Question 8: Let n and k be integers with $1 \leq n \leq k \leq 2n$. In this exercise, you will prove that

$$\sum_{i=k-n}^{n} \binom{k}{i} \binom{2n-k}{n-i} = \binom{2n}{n}. \tag{2}$$

Jim is working on the first assignment for the course COMP 4999 (Computational Aspects of Growing Cannabis). There are $2n$ questions on this assignment and each of them is worth 1 mark. Two minutes before the deadline, Jim has completed the first k questions. Jim is very smart and all answers to these k questions are correct. Jim knows that the instructor, Professor Mary Juana, does not accept late submissions. Because of this, Jim leaves the last $2n - k$ questions blank and hands in his assignment.

Tri is a TA for this course[1]. Since Tri is lazy, he does not want to mark all questions. Instead, he chooses a uniformly random subset of n questions out of the $2n$ questions, and only marks the n chosen questions. For each correct answer, Tri gives 2 marks, whereas he gives 0 marks for each wrong (or blank) answer.

For each integer $i \geq 0$, define the event

$$A_i = \text{“Jim receives exactly } 2i \text{ marks for his assignment”}.$$

- Determine the value of the summation $\sum_i \Pr(A_i)$. Explain your answer in plain English.

- Determine all values of i for which the event A_i is non-empty. For each such value i, determine $\Pr(A_i)$.

- Prove that (2) holds by combining the results of the previous parts.

Solution: We start with the first part of the question. The events A_i are pairwise disjoint and, together, cover all possible cases than can occur. Therefore,

$$\sum_i \Pr(A_i) = \Pr\left(\bigcup_i A_i\right) = \Pr(S) = 1,$$

where S is the sample space. Note that for this, we do not have to know what the actual samples space is.

For the second part, note that $k \geq n$.

1. The event A_i occurs if and only if Tri chooses exactly i questions from questions $1, 2, \ldots, n, \ldots, k$, and, therefore, Tri chooses exactly $n - i$ questions from questions $k + 1, k + 2, \ldots, n$.

2. Since Tri chooses a total of n questions, we must have $i \leq n$.

[1]Tri is also a TA for COMP 2804

3. The number of blank questions is equal to $2n - k$. Therefore, $n - i \leq 2n - k$, which is equivalent to $k - n \leq i$.

4. We conclude that the event A_i is non-empty if and only if $k - n \leq i \leq n$.

5. Let i be an integer with $k - n \leq i \leq n$. How many ways are there for Tri to make the event A_i occur:

   (a) There are $\binom{k}{i}$ ways to choose i questions from the first k questions.

   (b) There are $\binom{2n-k}{n-i}$ ways to choose $n - i$ questions from the last $2n - k$ questions.

   (c) By the Product Rule, the number of ways for which the event A_i occurs is equal to
   $$\binom{k}{i} \cdot \binom{2n-k}{n-i}.$$

   (d) The total number of ways for Tri to choose n questions from all $2n$ questions is equal to $\binom{2n}{n}$.

   (e) We conclude that
   $$\Pr(A_i) = \frac{\binom{k}{i} \cdot \binom{2n-k}{n-i}}{\binom{2n}{n}}.$$

For the third part: We have seen above that $\sum_i \Pr(A_i) = 1$. In this summation, we only have to consider integers i for which $k - n \leq i \leq n$. Therefore,

$$\sum_{i=k-n}^{n} \Pr(A_i) = \sum_{i=k-n}^{n} \frac{\binom{k}{i} \cdot \binom{2n-k}{n-i}}{\binom{2n}{n}} = 1.$$

If we multiply both sides by $\binom{2n}{n}$, then we get

$$\sum_{i=k-n}^{n} \binom{k}{i}\binom{2n-k}{n-i} = \binom{2n}{n}.$$

Note: This is a variant of the Vandermonde identity that we have seen in class.

Question 9: Let a and z be integers with $a > z \geq 1$, and let p be a real number with $0 < p < 1$. Alexa[2] and Zoltan[3] play a game consisting of several rounds. In each round,

1. Alexa receives a points with probability p and 0 points with probability $1 - p$,

2. Zoltan receives z points (with probability 1).

We assume that the results of different rounds are independent.

[2] your friendly TA

[3] another friendly TA

- Define the event

$$A = \text{“in one round, Alexa receives more points than Zoltan”}.$$

  We say that Alexa is a *better player* than Zoltan, if $\Pr(A) > 1/2$.

  For which values of p is Alexa a better player than Zoltan?

- Assume that $a = 3$, $z = 2$, and p is chosen such that $p > 1/2$ and $p^2 < 1/2$. (For example, $p = (\sqrt{5} - 1)/2$.)

  - Is Alexa a better player than Zoltan?
  - Alexa and Zoltan play a game consisting of two rounds. We consider the total number of points that each player wins during these two rounds. Define the event

    $$B = \text{“in two rounds, Alexa receives more points than Zoltan”}.$$

    Prove that $\Pr(B) < 1/2$. (This seems to suggest that Zoltan is a better player than Alexa.)

- Let n be a large integer, and assume that $a = n + 1$, $z = n$, and p is chosen very close to (but less than) 1. (For example, $n = 500$ and $p = 0.99$.)

  - Is Alexa a better player than Zoltan?
  - Alexa and Zoltan play a game consisting of n rounds. We consider the total number of points that each player wins during these n rounds. Define the event

    $$C = \text{“in } n \text{ rounds, Alexa receives more points than Zoltan”}.$$

    Prove that $\Pr(C) = p^n$. (If $n = 500$ and $p = 0.99$, then $p^n \approx 0.0066$. This seems to suggest that Zoltan is a *much* better player than Alexa.)

Solution: We start with the first part: Since in one round, Zoltan always receives z points, and since $a > z$, we have

$$A \text{ if and only if in one round, Alexa receives } a \text{ points.}$$

This implies that $\Pr(A) = p$. Thus, Alexa is a better player than Zoltan if and only if $p > 1/2$.

For the second part: Since $p > 1/2$, Alexa is a better player than Zoltan. Consider the different possibilities in two rounds:

1. The number of points received by Alexa can be $3 + 3 = 6$, or $3 + 0 = 3$, or $0 + 3 = 3$, or $0 + 0 = 0$.

2. The number of points received by Zoltan is always equal to $2 + 2 = 4$.

Thus,

$$B \text{ if and only if in two rounds, Alexa receives } 3+3=6 \text{ points.}$$

This gives

$$\begin{aligned}
\Pr(B) &= \Pr(\text{ 3 points in first round} \wedge \text{3 points in second round}) \\
&= \Pr(\text{ 3 points in first round }) \cdot \Pr(\text{ 3 points in second round}) \\
&= p \cdot p \\
&= p^2 \\
&< 1/2.
\end{aligned}$$

For the third part: Since $p > 1/2$, Alexa is a better player than Zoltan.

We first prove that the event C occurs if and only if Alexa receives $n+1$ points in each of the n rounds:

1. Assume that Alexa receives $n+1$ points in each of the n rounds. Then Alexa receives a total of $n(n+1) = n^2 + n$ points. The number of points that Zoltan receives in n rounds is $n \cdot n = n^2$. Since $n^2 + n > n^2$, event C occurs.

2. Now assume that Alexa does not receive $n+1$ points in each of the n rounds. Then there is at least one round, in which Alexa receives 0 points. Thus, the total number of points that Alexa receives in n rounds is at most $(n-1)(n+1) = n^2 - 1$. The number of points that Zoltan receives in n rounds is $n \cdot n = n^2$. Since $n^2 - 1 < n^2$, event C does not occur.

This gives

$$\begin{aligned}
\Pr(C) &= \Pr(\text{ Alexa receives } n+1 \text{ points in each of the } n \text{ rounds }) \\
&= \prod_{i=1}^{n} \Pr(\text{ Alexa receives } n+1 \text{ points in round } i) \\
&= \prod_{i=1}^{n} p \\
&= p^n.
\end{aligned}$$

Note: In one round, Alexa is more likely to win. For large values of n, in n rounds, Zoltan will win with probability $1 - p^n$, which is very close to 1. In other words, Alexa is very unlikely to win a game consisting of many rounds.