COMP 2804 — Assignment 2

Due: Wednesday February 15, before 4:30pm, in the course drop box in Herzberg 3115.

Assignment Policy: Late assignments will not be accepted. Students are encouraged to collaborate on assignments, but at the level of discussion only. When writing the solutions, they should do so in their own words. Past experience has shown conclusively that those who do not put adequate effort into the assignments do not learn the material and have a probability near 1 of doing poorly on the exams.

Important note: When writing your solutions, you must follow the guidelines below.

- The answers should be concise, clear and neat.
- All answers must be justified.
- Assignments should be stapled or placed in an unsealed envelope.

Substantial departures from the above guidelines will not be graded.

Question 1: On the first page of your assignment, write your name and student number.

Question 2: The function $f : \mathbb{N} \rightarrow \mathbb{N}$ is defined by

$$\begin{aligned} f(0) &= 0, \\ f(1) &= 0, \\ f(n) &= f(n-2) + 2^{n-1} \quad \text{if } n \geq 2. \end{aligned}$$

- Prove that for every even integer $n \geq 0$,

$$f(n) = \frac{2^{n+1} - 2}{3}.$$

- Prove that for every odd integer $n \geq 1$,

$$f(n) = \frac{2^{n+1} - 4}{3}.$$

Question 3: The function $f : \mathbb{N}^2 \rightarrow \mathbb{N}$ is defined by

$$\begin{aligned} f(0, n) &= 2n && \text{if } n \geq 0, \\ f(m, 0) &= 0 && \text{if } m \geq 1, \\ f(m, 1) &= 2 && \text{if } m \geq 1, \\ f(m, n) &= f(m-1, f(m, n-1)) && \text{if } m \geq 1 \text{ and } n \geq 2. \end{aligned}$$

- Determine $f(2, 2)$.

- Determine $f(1,n)$ for $n \geq 1$.
- Determine $f(3,3)$.

Question 4: A *block* in a bitstring is a maximal sequence of consecutive 1's. For example, the bitstring 11011101000 contains the three blocks

$$\underbrace{11}\,0\,\underbrace{111}\,0\,\underbrace{1}\,000.$$

A bitstring is called *awesome*, if each of its blocks has an even length. Thus, the bitstring above is not awesome, whereas both bitstrings 00011011110 and 0000000 are awesome.

For any integer $n \geq 1$, let A_n denote the number of awesome bitstrings of length n.

- Determine A_1, A_2, A_3, and A_4.
- Determine the value of A_n, i.e., express A_n in terms of numbers that we have seen in class.

  Hint: Derive a recurrence relation.

Question 5: The Fibonacci numbers are defined as follows: $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$.

In class, we have seen that for any $m \geq 1$, the number of 00-free bitstrings of length m is equal to f_{m+2}.

- Let $n \geq 2$ be an integer. What is the number of 00-free bitstrings of length $2n-1$ for which the bit in the middle position is equal to 1?
- Let $n \geq 3$ be an integer. What is the number of 00-free bitstrings of length $2n-1$ for which the bit in the middle position is equal to 0?
- Use the previous results to prove that for any integer $n \geq 3$,

$$f_{2n+1} = f_n^2 + f_{n+1}^2.$$

Question 6: Elisa Kazan[1] loves to drink cider. During the weekend, Elisa goes to the pub and runs the following recursive algorithm, which takes as input an integer $n \geq 0$:

[1] President of the Carleton Computer Science Society

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Algorithm ElisaGoesToThePub(n):

    if n = 0
    then order Fibonachos
    else if n is even
        then ElisaGoesToThePub(n/2);
             drink n^2/2 pints of cider;
             ElisaGoesToThePub(n/2)
        else for i = 1 to 4
             do ElisaGoesToThePub((n − 1)/2);
                drink (n − 1)/2 pints of cider
             endfor;
             drink 1 pint of cider
        endif
    endif
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For $n \geq 0$, let $C(n)$ be the number of pints of cider that Elisa drinks when running algorithm ElisaGoesToThePub(n). Determine the value of $C(n)$.

Question 7: In the fall term of 2015, Nick[2] took COMP 2804. Nick was always sitting in the back of the classroom and spent most of his time eating bananas. Nick uses the following banana-buying-scheme:

- At the start of week 0, there are 2 bananas in Nick's fridge.
- For any integer $n \geq 0$, Nick does the following during week n:
  - At the start of week n, Nick determines the number of bananas in his fridge and stores this number in a variable x.
  - Nick goes to Jim's Banana Empire, buys x bananas, and puts them in his fridge.
  - Nick takes $n + 1$ bananas out of his fridge and eats them during week n.

For any integer $n \geq 0$, let $B(n)$ be the number of bananas in Nick's fridge at the start of week n. Determine the value of $B(n)$.

[2] your friendly TA