COMP 2804 — Assignment 3

Due: Wednesday March 22, before 4:30pm, in the course drop box in Herzberg 3115.

Assignment Policy: Late assignments will not be accepted. Students are encouraged to collaborate on assignments, but at the level of discussion only. When writing the solutions, they should do so in their own words. Past experience has shown conclusively that those who do not put adequate effort into the assignments do not learn the material and have a probability near 1 of doing poorly on the exams.

Important note: When writing your solutions, you must follow the guidelines below.

- The answers should be concise, clear and neat.
- When presenting proofs, every step should be justified.
- Assignments should be stapled or placed in an unsealed envelope.

Substantial departures from the above guidelines will not be graded.

Question 1: On the first page of your assignment, write your name and student number.

Question 2: You are given a red coin and a blue coin. Both coins have the number 1 on one side and the number 2 on the other side. You flip both coins once (independently of each other) and take the sum of the two results. Define the events

$$\begin{aligned} A &= \text{“the sum of the results equal 2”}, \\ B &= \text{“the sum of the results equals 3”}, \\ C &= \text{“the sum of the results equals 4”}. \end{aligned}$$

- Assume both coins are fair. Determine $\Pr(A)$, $\Pr(B)$, and $\Pr(C)$. Show your work.
- Let p and q be real numbers with $0 < p < 1$ and $0 < q < 1$. Assume the red coin comes up "1" with probability p and the blue coin comes up "1" with probability q. Is it possible to choose p and q such that

  $$\Pr(A) = \Pr(B) = \Pr(C)?$$

  As always, justify your answer.

Question 3: Elisa and Nick go to Tan Tran's Darts Bar. When Elisa throws a dart, she hits the dartboard with probability p. When Nick throws a dart, he hits the dartboard with probability q. Here, p and q are real numbers with $0 < p < 1$ and $0 < q < 1$. Elisa and Nick throw one dart each, independently of each other. Define the events

$$\begin{aligned} E &= \text{“Elisa's dart hits the dartboard”}, \\ N &= \text{“Nick's dart hits the dartboard”}. \end{aligned}$$

Use the formal definition of conditional probability to determine

$$\Pr(E \mid E \cup N)$$

and

$$\Pr(E \cap N \mid E \cup N).$$

Show your work.

Question 4: Let $n \geq 4$ be an integer. Consider a uniformly random permutation of $\{1, 2, \dots, n\}$ and define the events

$$A = \text{"1 and 2 are next to each other, with 1 to the left of 2, or 4 and 3 are next to each other, with 4 to the left of 3"}$$

and

$$B = \text{"1 and 2 are next to each other, with 1 to the left of 2, or 2 and 3 are next to each other, with 2 to the left of 3".}$$

- Determine $\Pr(A)$ and $\Pr(B)$.

  Before you answer this question, spend a few seconds on guessing which probability is larger.

Question 5: Let A be an event in some probability space $(S, \Pr)$. You are given that the events A and A are independent[1]. Determine $\Pr(A)$. Show your work.

Question 6: Three people P_1, P_2, and P_3 are in a dark room. Each person has a bag containing one red hat and one blue hat. Each person chooses a uniformly random hat from her bag and puts it on her head. Afterwards, the lights are turned on.

Each person does not know the color of her hat, but can see the colors of the other two hats. Each person P_i can do one of the following:

- Person P_i announces "my hat is red".
- Person P_i announces "my hat is blue".
- Person P_i says "I pass".

The game is a *success* if at least one person announces the correct color of her hat and no person announces the wrong color of her hat. (If a person passes, then she does not announce any color.)

- Assume person P_1 announces "my hat is red" and both P_2 and P_3 pass. Define the event

  $$A = \text{"the game is a success."}$$

  Determine $\Pr(A)$. Show your work.

[1] This is not a typo.

- Assume each person P_i does the following:
  - If the two hats that P_i sees have different colors, then P_i passes.
  - If the two hats that P_i sees are both red, then P_i announces "my hat is blue".
  - If the two hats that P_i sees are both blue, then P_i announces "my hat is red".

  Define the event

  $$B = \text{"the game is a success."}$$

  Determine $\Pr(B)$. Show your work.

Question 7: Let $n \geq 0$ be an integer. In this question, you will prove that

$$\sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k} = \frac{1}{n+1} \left(2^{n+1} - 1\right). \tag{1}$$

There are $n+1$ students in Carleton's Computer Science program. We denote these students by $P_1, P_2, \ldots, P_{n+1}$. We play the following game:

1. We choose a uniformly random subset X of $\{P_1, P_2, \ldots, P_{n+1}\}$.
2. (a) If $X \neq \emptyset$, then we choose a uniformly random student in X. The chosen student wins a six-pack of cider.

   (b) If $X = \emptyset$, then nobody wins the six-pack.

The random choices made are independent of each other.

- Define the event

  $$A_0 = \text{"nobody wins the six-pack"}.$$

  Determine $\Pr(A_0)$. Justify your answer.

- For each $i = 1, 2, \ldots, n+1$, define the event

  $$A_i = \text{"student } P_i \text{ wins the six-pack"}.$$

  Explain in plain English, and in at most two sentences, why

  $$\Pr(A_1) = \Pr(A_2) = \ldots = \Pr(A_{n+1}).$$

- Prove that

  $$\Pr(A_1) = \frac{1 - 1/2^{n+1}}{n+1}.$$

- For each k with $0 \le k \le n$, define the event

$$B_k = \text{“}X \text{ has size } k+1 \text{ and } P_1 \text{ wins the six-pack”}.$$

Prove that

$$\Pr(B_k) = \frac{\binom{n}{k}}{2^{n+1}} \cdot \frac{1}{k+1}.$$

- Express the event A_1 in terms of the events $B_0, B_1, \ldots, B_n$.

- Prove that (1) holds by combining the results of the previous parts.

Question 8: You roll a fair die once. Define the events

$$\begin{aligned} A &= \text{“the result is even”}, \\ B &= \text{“the result is odd”}, \\ C &= \text{“the result is at most 4”}. \end{aligned}$$

For each of the following questions, justify your answer.

- Are the events A and B independent?

- Are the events A and C independent?

- Are the events B and C independent?

Question 9: Let n be a large power of two (thus, $\log n$ is an integer). Consider a binary string $s = s_1 s_2 \ldots s_n$, where each bit s_i is 0 with probability 1/2, and 1 with probability 1/2, independently of the other bits.

A *run of length* k is a substring of length k, all of whose bits are equal. In class, we have seen that it is very likely that the bitstring s contains a run of length at least $\log n - 2 \log \log n$. In this exercise, you will prove that it is very unlikely that s contains a run of length more than $2 \log n$.

- Let k be an integer with $1 \le k \le n$. Define the event

$$A = \text{“the bitstring } s \text{ contains a run of length at least } k\text{”}.$$

For each i with $1 \le i \le n - k + 1$, define the event

$$A_i = \text{“the substring } s_i s_{i+1} \ldots s_{i+k-1} \text{ is a run”}.$$

Use the Union Bound (Lemma 5.3.5 on page 135 of the textbook) to prove that

$$\Pr(A) \le \frac{n-k+1}{2^{k-1}}.$$

- Let $k = 2 \log n$. Prove that

$$\Pr(A) \le 2/n.$$