COMP 2804 — Assignment 4

Due: Wednesday April 5, before 4:30pm, in the course drop box in Herzberg 3115.

Assignment Policy: Late assignments will not be accepted. Students are encouraged to collaborate on assignments, but at the level of discussion only. When writing the solutions, they should do so in their own words. Past experience has shown conclusively that those who do not put adequate effort into the assignments do not learn the material and have a probability near 1 of doing poorly on the exams.

Important note: When writing your solutions, you must follow the guidelines below.

- The answers should be concise, clear and neat.

- When presenting proofs, every step should be justified.

- Assignments should be stapled or placed in an unsealed envelope.

Substantial departures from the above guidelines will not be graded.

Question 1: On the first page of your assignment, write your name and student number.

Question 2: Two players P_1 and P_2 play a game in which they take turns flipping, independently, a fair coin: First P_1 flips the coin, then P_2 flips the coin, then P_1 flips the coin, then P_2 flips the coin, etc. The game ends as soon as the sequence of coin flips contains either HH or TT. The player who flips the coin for the last time is the winner of the game. For example, if the sequence of coin flips is $HTHTHH$, then P_2 wins the game.
 Determine the probability that P_1 wins the game. Show your work.

Question 3: Assume we flip a fair coin twice, independently of each other. Define the following random variables:

$$
\begin{aligned}
X &= \text{the number of heads,} \\
Y &= \text{the number of tails,} \\
Z &= \text{the number of heads times the number of tails.}
\end{aligned}
$$

- Determine the expected values of these three random variables. Show your work.

- Are X and Y independent random variables? Justify your answer.

- Are X and Z independent random variables? Justify your answer.

- Are Y and Z independent random variables? Justify your answer.

Question 4: Consider the sample space $S = \{1, 2, 3 \ldots, 10\}$. We choose a uniformly random element x in S. Define the following random variables:

$$X = \begin{cases} 0 & \text{if } x \in \{1, 2\}, \\ 1 & \text{if } x \in \{3, 4, 5, 6\}, \\ 2 & \text{if } x \in \{7, 8, 9, 10\} \end{cases}$$

and

$$Y = \begin{cases} 0 & \text{if } x \text{ is even}, \\ 1 & \text{if } x \text{ is odd}. \end{cases}$$

Are X and Y independent random variables? Justify your answer.

Question 5: In order to attract more customers, the Hyacintho Cactus Bar and Grill in downtown Ottawa organizes a game night, hosted by their star employee Tan Tran.

After paying \$26, a player gets two questions P and Q. If the player gives the correct answer to question P, this player wins \$30; if the player gives the correct answer to question Q, this player wins \$60. A player can choose between the following two options:

1. Start with question P. In this case, the player is allowed to answer question Q only if the answer to question P is correct.

2. Start with question Q. In this case, the player is allowed to answer question P only if the answer to question Q is correct.

Elisa decides to play this game. The probability that Elisa correctly answers question P is equal to $1/2$, whereas she correctly answers question Q with probability $1/3$. The events of correctly answering are independent.

- Assume Elisa chooses the first option. Define the random variable X to be the amount of money that Elisa wins (this includes the \$26 that she has to pay in order to play the game). Determine the expected value $\mathbb{E}(X)$. Show your work.

- Assume Elisa chooses the second option. Define the random variable Y to be the amount of money that Elisa wins (this includes the \$26 that she has to pay in order to play the game). Determine the expected value $\mathbb{E}(Y)$. Show your work.

Question 6: Consider a uniformly random permutation $a_1, a_2, \ldots, a_n$ of the set $\{1, 2, \ldots, n\}$. Define the random variable X to be the number of ordered pairs (i, j) with $1 \leq i < j \leq n$ for which $a_i = j$ and $a_j = i$. Determine the expected value $\mathbb{E}(X)$ of X. Show your work.
   Hint: Use indicator random variables.

Question 7: Nick[1] wants to know how many students cheat on the assignments. One approach is to ask every student "Did you cheat?". This obviously does not work, because

---
[1]your friendly TA

every student will answer "I did not cheat". Instead, Nick uses the following ingenious scheme, which gives a reasonable estimate of the number of cheaters, without identifying them.

We denote the students by $S_1, S_2, \ldots, S_n$. Let k denote the number of cheaters. Nick knows the value of n, but he does not know the value of k.

For each i with $1 \leq i \leq n$, Nick does the following:

1. Nick meets student S_i and asks "Did you cheat?".

2. Student S_i flips a fair coin twice, independently of each other; S_i does not show the results of the coin flips to Nick.

   (a) If the coin flips are HH or HT, then S_i is honest in answering the question: If S_i is a cheater, then he answers "I cheated"; otherwise, he answers "I did not cheat".

   (b) If the coin flips are TH, then S_i answers "I cheated".

   (c) If the coin flips are TT, then S_i answers "I did not cheat".

- Define the random variable X to be the number of students who answer "I cheated". Determine the expected value $\mathbb{E}(X)$ of X.

  Hint: For each i, use an indicator random variable X_i which indicates whether or not S_i answers "I cheated". If S_i is a cheater, what is $\mathbb{E}(X_i)$? If S_i is not a cheater, what is $\mathbb{E}(X_i)$?

- Define the random variable
$$Y = 2X - n/2.$$

  Prove that $\mathbb{E}(Y) = k$. In words, the expected value of Y is equal to the number of cheaters.