COMP 2804 — Solutions Assignment 1

Question 1: On the first page of your assignment, write your name and student number.

Solution:

- Name: Daniel Alfredsson

- Student number: 11

Question 2: A password is a string of ten characters, where each character is a lowercase letter, a digit, or one of the eight special characters !, @, #, $, %, &, (, and).

A password is called *awesome*, if it contains at least one digit or at least one special character. Determine the number of awesome passwords.

Solution: The number of possibilities for one character is $26 + 10 + 8 = 44$. This implies that the total number of possible passwords is equal to 44^{10}: There are 10 characters in one password, and 44 choices for each character.

A password is not awesome if it does not contain any digit and does not contain any special character. This means that an un-awesome password contains lowercase letters only: There are 10 characters, and 26 choices for each character. Thus, the number of un-awesome passwords is equal to 26^{10}.

We conclude that the number of awesome passwords is equal to

$$44^{10} - 26^{10}.$$

Question 3: Determine the number of functions

$$f : \{1, 2, 3, 4\} \rightarrow \{a, b, c, \ldots, z\},$$

such that $f(1) = f(2)$, or $f(3) = f(4)$, or $f(1) \neq f(3)$.

Solution: We define the following three sets:

- A is the set of all functions f such that $f(1) = f(2)$,

- B is the set of all functions f such that $f(3) = f(4)$,

- C is the set of all functions f such that $f(1) \neq f(3)$.

Our goal is to compute the size of the set $A \cup B \cup C$. By the inclusion-exclusion principle, we have

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

We are going to determine each term on the right-hand side:

- $|A|$:

  - Choose $f(1)$. This can be done in 26 ways.
  - For $f(2)$, take the same value as we took for $f(1)$. This can be done in 1 way.
  - Choose $f(3)$. This can be done in 26 ways.
  - Choose $f(4)$. This can be done in 26 ways.

  By the Product Rule,
  $$|A| = 26 \cdot 1 \cdot 26 \cdot 26 = 26^3.$$

- $|B|$: By the same reasoning, we get
  $$|B| = 26^3.$$

- $|C|$:

  - Choose $f(1)$. This can be done in 26 ways.
  - Choose $f(2)$. This can be done in 26 ways.
  - Choose $f(3)$. This can be done in 25 ways, because $f(3)$ cannot be equal to $f(1)$.
  - Choose $f(4)$. This can be done in 26 ways.

  By the Product Rule,
  $$|C| = 26 \cdot 26 \cdot 25 \cdot 26 = 25 \cdot 26^3.$$

- $|A \cap B|$:

  - Choose $f(1)$. This can be done in 26 ways.
  - For $f(2)$, take the same value as we took for $f(1)$. This can be done in 1 way.
  - Choose $f(3)$. This can be done in 26 ways.
  - For $f(4)$, take the same value as we took for $f(3)$. This can be done in 1 way.

  By the Product Rule,
  $$|A \cap B| = 26 \cdot 1 \cdot 26 \cdot 1 = 26^2.$$

- $|A \cap C|$:

  - Choose $f(1)$. This can be done in 26 ways.
  - For $f(2)$, take the same value as we took for $f(1)$. This can be done in 1 way.
  - Choose $f(3)$. This can be done in 25 ways, because $f(3)$ cannot be equal to $f(1)$.
  - Choose $f(4)$. This can be done in 26 ways.

By the Product Rule,
$$|A \cap C| = 26 \cdot 1 \cdot 25 \cdot 26 = 25 \cdot 26^2.$$

- $|B \cap C|$:

  – Choose $f(1)$. This can be done in 26 ways.
  – Choose $f(2)$. This can be done in 26 ways.
  – Choose $f(3)$. This can be done in 25 ways, because $f(3)$ cannot be equal to $f(1)$.
  – For $f(4)$, take the same value as we took for $f(3)$. This can be done in 1 way.

  By the Product Rule,
  $$|B \cap C| = 26 \cdot 26 \cdot 25 \cdot 1 = 25 \cdot 26^2.$$

- $|A \cap B \cap C|$:

  – Choose $f(1)$. This can be done in 26 ways.
  – For $f(2)$, take the same value as we took for $f(1)$. This can be done in 1 way.
  – Choose $f(3)$. This can be done in 25 ways, because $f(3)$ cannot be equal to $f(1)$.
  – For $f(4)$, take the same value as we took for $f(3)$. This can be done in 1 way.

  By the Product Rule,
  $$|A \cap B \cap C| = 26 \cdot 1 \cdot 25 \cdot 1 = 25 \cdot 26.$$

We conclude that
$$|A \cup B \cup C| = \text{ (plug in the numbers) } = 440,726.$$

Question 4: Let $n \geq 1$ be an integer.

- Assume that n is odd. Determine the number of bitstrings of length n that contain more 0's than 1's. Justify your answer in plain English and at most three sentences.

  Hint: Symmetry.

  Solution: By symmetry in 0's and 1's, the number of bitstrings of length n that contain more 0's than 1's is equal to the number of bitstrings of length n that contain more 1's than 0's. In total, there are 2^n bitstrings of length n. Exactly half of them have more 0's than 1's. Thus, the answer is 2^{n-1}.

- Assume that n is even.

– Determine the number of bitstrings of length n in which the number of 0's is equal to the number of 1's.

Solution: The number of 0's is equal to $n/2$. Thus, the answer is

$$\binom{n}{n/2}.$$

– Determine the number of bitstrings of length n that contain strictly more 0's than 1's.

Solution: Let x be the number of bitstrings of length n that contain strictly more 0's than 1's. By symmetry, x is also equal to the number of bitstrings of length n that contain strictly more 1's than 0's.

In each bitstring, the number of 0's is strictly more than, or strictly less than, or equal to the number of 1's. These three ORs are mutually exclusive. Therefore,

$$2x + \binom{n}{n/2} = 2^n.$$

We conclude that

$$x = 2^{n-1} - \frac{1}{2}\binom{n}{n/2}.$$

– Argue that the binomial coefficient

$$\binom{n}{n/2}$$

is an even integer.

Solution: We have seen above that

$$\binom{n}{n/2} = 2^n - 2x.$$

Since the right-hand side is even, the left-hand side is even as well.

Question 5: Let m, n, k, and ℓ be integers such that $m \geq 1$, $n \geq 1$, $1 \leq \ell \leq k \leq \ell + m$, and $\ell \leq n$.

After a week of hard work, Elisa Kazan[1] goes to her neighborhood pub. This pub has m different types of beer and n different types of cider on tap. Elisa decides to order k pints: At most one pint of each type, and exactly ℓ pints of cider. Determine the number of ways in which Elisa can order these k pints. The order in which Elisa orders matters.

Solution: We are going to use the Product Rule:

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[1]President of the Carleton Computer Science Society

- Choose ℓ pints of cider. There are $\binom{n}{\ell}$ ways to do this.

- Choose $k - \ell$ pints of beer . There are $\binom{m}{k-\ell}$ ways to do this.

- Put the k pints in order. There are $k!$ ways to do this.

By the Product Rule, the final answer is

$$\binom{n}{\ell}\binom{m}{k-\ell}k!.$$

Question 6: Nick is not only your friendly TA, he also has a part-time job in a grocery store. This store sells n different types of India Pale Ale (IPA) and n different types of wheat beer, where $n \geq 2$ is an integer. Prove that

$$\binom{2n}{2} = 2\binom{n}{2} + n^2,$$

by counting, in two different ways, the number of ways to choose two different types of beer.

Solution: Nick's store sells $2n$ different types of beer. Thus, the number of ways to choose two different beers is equal to

$$\binom{2n}{2}. \tag{1}$$

Alternatively, we can count as follows: To choose two different types of beer,

- we choose two different types of IPA; there are $\binom{n}{2}$ ways to do this, or

- we choose two different types of wheat beer; there are $\binom{n}{2}$ ways to do this, or

- we choose one IPA and one wheat beer; there are $n \cdot n = n^2$ ways to do this.

Thus, the number of ways to choose two different beers is equal to

$$\binom{n}{2} + \binom{n}{2} + n^2. \tag{2}$$

Since (1) and (2) must be equal (because they count the same things), the answer is complete.

Question 7: A string consisting of characters is called *cool*, if exactly one character in the string is equal to the letter x and each other character is a digit. Let $n \geq 1$ be an integer.

- Determine the number of cool strings of length n.

    Solution: To obtain a cool string of length n, we do the following:

– Choose one position for the letter x. There are n ways to do this.

– In each of the remaining $n - 1$ positions, write a digit. There are 10^{n-1} ways to do this.

By the Product Rule, the total number of cool strings of length n is equal to

$$n \cdot 10^{n-1}. \tag{3}$$

- Let k be an integer with $1 \le k \le n$. Determine the number of cool strings of length n that contain exactly $n - k$ many 0's.

  Solution: The strings that we want to count contain one letter x, $n - k$ many 0's, and $k - 1$ non-zero digits.

  To obtain such a string of length n, we do the following:

  – Choose k positions. There are $\binom{n}{k}$ ways to do this.

    * Choose one of these k positions and write the letter x in that position. There are k ways to do this.

    * In each of the other $k - 1$ positions, write a non-zero digit. There are 9^{k-1} ways to do this.

  – In each of the remaining $n - k$ positions, write a 0. There is one way to do this.

  By the Product Rule, the total number of cool strings of length n with exactly $n - k$ many 0's is equal to

  $$\binom{n}{k} \cdot k \cdot 9^{k-1}. \tag{4}$$

- Use the above two results to prove that

$$\sum_{k=1}^{n} k \binom{n}{k} 9^{k-1} = n \cdot 10^{n-1}.$$

  Solution: (3) gives the total number of cool strings of length n. To count these strings in a different way, we are going to divide them into pairwise disjoint groups: In group k, we consider all cool strings of length n that have exactly $n - k$ many 0's.

  The number of 0's in any cool string of length n can be any number from the set $\{0, 1, 2, \ldots, n - 1\}$. (It cannot be n because there must be one symbol x.) If we use $n - k$ to denote the number of 0's, then $n - k$ can be any number in $\{0, 1, 2, \ldots, n-1\}$. This means that k can be any number in $\{1, 2, \ldots, n\}$. The number of strings in group k is given by (4). If we take their sum (where k runs from 1 to n), then we have counted every cool string of length n exactly once.

Question 8: Determine the number of elements x in the set $\{1, 2, 3, \ldots, 99999\}$ for which the sum of the digits in the decimal representation of x is equal to 8. An example of such an element x is 3041.

You may use any result that was proven in class.

Solution: Each element x of the set $\{1, 2, 3, \ldots, 99999\}$ has at most five digits, when written in decimal. If it has fewer digits, then we add 0's at the left, so that it has exactly five digits. For example, instead of 3041, we write 03041. Similarly, instead of 1, we write 00001.

Consider an element x of the set $\{1, 2, 3, \ldots, 99999\}$. Consider its decimal representation:

$$x_5 x_4 x_3 x_2 x_1.$$

Then we include x in our count, if

$$x_1 + x_2 + x_3 + x_4 + x_5 = 8.$$

Thus, we have to count the number of solutions to this equation, where each x_i is an integer that is at least 0. Note that in any such solution, $0 \leq x_i \leq 8$ and, thus, x_i is a digit in $\{0, 1, 2, \ldots, 9\}$. We have seen in class that the answer is

$$\binom{12}{4}.$$

NOTE: If the requirement would have been that the sum of the digits is equal to any fixed number strictly greater than 9, then this approach does not work.

Question 9: Let $n \geq 2$ be an integer and let $G = (V, E)$ be a graph whose vertex set V has size n and whose edge set E is non-empty. The degree of any vertex u is defined to be the number of edges in E that contain u as a vertex.

- Prove that there exist at least two vertices in G that have the same degree.

  Hint: Consider the cases when G is connected and G is not connected separately. In each case, apply the Pigeonhole Principle. Alternatively, consider a vertex of maximum degree together with its adjacent vertices and, again, apply the Pigeonhole Principle.

Solution: The degree of any vertex is an element of the set $\{0, 1, 2, \ldots, n-1\}$; this set has size n. Since there are n vertices, there are n degrees, each one belonging to a set of size n. This does not help us!

We consider two cases:

- Assume the graph G is connected.

  Then no vertex has degree 0: If there is a vertex of degree 0, then the graph is not connected.

Thus, the degree of any vertex is an element of the set $\{1, 2, \ldots, n - 1\}$; this set has size $n - 1$. Since there are n vertices, there are n degrees, each one belonging to a set of size $n - 1$. By the Pigeonhole Principle, there must be two vertices that have the same degree.

- Assume the graph G is not connected.

  Then no vertex has degree $n - 1$: If there is a vertex of degree $n - 1$, then this vertex is connected to all other vertices and, thus, the graph is connected.

  Thus, the degree of any vertex is an element of the set $\{0, 1, 2, \ldots, n - 2\}$; this set has size $n - 1$. Since there are n vertices, there are n degrees, each one belonging to a set of size $n - 1$. By the Pigeonhole Principle, there must be two vertices that have the same degree.

  Alternatively, consider a connected component G' of G that has at least two vertices. (There is such a component, because the edge set of G is non-empty.) Since G' is connected, it has two vertices u and v of the same degree (we have seen above that this is the case). In G, u and v have the same degree as well.

Here is a second solution: Consider a vertex u of maximum degree. Let d be the degree of u. Consider u, together with its d adjacent vertices. This gives $d + 1$ vertices. The degree of each of these vertices is an element of the set $\{1, 2, \ldots, d\}$; this set has size d. Thus, we have $d + 1$ vertices, they have $d + 1$ degrees, each one belonging to a set of size d. By the Pigeonhole Principle, there must be two vertices among these $d + 1$ vertices that have the same degree.