COMP 2804 — Solutions Assignment 3

Question 1: On the first page of your assignment, write your name and student number.

Solution:

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- Student number: 14

Question 2: You are given a red coin and a blue coin. Both coins have the number 1 on one side and the number 2 on the other side. You flip both coins once (independently of each other) and take the sum of the two results. Define the events

$$\begin{aligned} A &= \text{“the sum of the results equal 2”}, \\ B &= \text{“the sum of the results equals 3”}, \\ C &= \text{“the sum of the results equals 4”}. \end{aligned}$$

- Assume both coins are fair. Determine $\Pr(A)$, $\Pr(B)$, and $\Pr(C)$. Show your work.
- Let p and q be real numbers with $0 < p < 1$ and $0 < q < 1$. Assume the red coin comes up "1" with probability p and the blue coin comes up "1" with probability q. Is it possible to choose p and q such that

  $$\Pr(A) = \Pr(B) = \Pr(C)?$$

  As always, justify your answer.

Solution: We start with the case when both coins are fair. For each $i \in \{1, 2\}$, we define the events

$$\begin{aligned} R_i &= \text{“the result of the red coin is } i\text{”}, \\ B_i &= \text{“the result of the blue coin is } i\text{”}. \end{aligned}$$

Then $\Pr(A) = \Pr(R_1 \wedge B_1)$. Since the coin flips are independent, we get

$$\Pr(A) = \Pr(R_1 \wedge B_1) = \Pr(R_1) \cdot \Pr(B_1) = 1/2 \cdot 1/2 = 1/4.$$

By the same reasoning, we get

$$\Pr(C) = \Pr(R_2 \wedge B_2) = \Pr(R_2) \cdot \Pr(B_2) = 1/2 \cdot 1/2 = 1/4.$$

Since exactly one of the events A, B, and C is guaranteed to occur, we have

$$\Pr(A) + \Pr(B) + \Pr(C) = 1,$$

implying that $\Pr(B) = 1/2$. Alternatively, we have

$$\begin{aligned}
\Pr(B) &= \Pr\left((R_2 \wedge B_1) \vee (R_1 \wedge B_2)\right) \\
&= \Pr\left(R_2 \wedge B_1\right) + \Pr\left(R_1 \wedge B_2\right) \\
&= \Pr(R_2) \cdot \Pr(B_1) + \Pr(R_1) \cdot \Pr(B_2) \\
&= 1/2 \cdot 1/2 + 1/2 \cdot 1/2 \\
&= 1/2.
\end{aligned}$$

Next we do the second part of the question. As above, we have

$$\Pr(A) + \Pr(B) + \Pr(C) = 1.$$

Thus, if $\Pr(A) = \Pr(B) = \Pr(C)$, then each of them is equal to $1/3$.

As above, we get

$$\Pr(A) = pq = 1/3,$$

implying that

$$q = \frac{1}{3p}. \tag{1}$$

As above, we get

$$\Pr(C) = (1-p)(1-q) = 1 - p - q + pq = 1/3.$$

Since $pq = 1/3$, we get $1 - p - q + 1/3 = 1/3$, which is equivalent to

$$p = 1 - q. \tag{2}$$

If we combine (1) and (2), we get

$$p = 1 - \frac{1}{3p},$$

which is equivalent to

$$3p^2 = 3p - 1,$$

which is equivalent to

$$3p^2 - 3p + 1 = 0.$$

In highschool, you have learned that the equation $Ax^2 + Bx + C = 0$ has a solution in $\mathbb{R}$ if and only if $B^2 - 4AC \geq 0$. In our case, we have

$$B^2 - 4AC = (-3)^2 - 4 \cdot 3 \cdot 1 = -3 < 0.$$

We conclude that the equation $3p^2 - 3p + 1 = 0$ does not have a solution in $\mathbb{R}$. In other words, it is not possible to choose p and q such that $\Pr(A) = \Pr(B) = \Pr(C)$.

Question 3: Elisa and Nick go to Tan Tran's Darts Bar. When Elisa throws a dart, she hits the dartboard with probability p. When Nick throws a dart, he hits the dartboard with probability q. Here, p and q are real numbers with $0 < p < 1$ and $0 < q < 1$. Elisa and Nick throw one dart each, independently of each other. Define the events

$$\begin{aligned} E &= \text{"Elisa's dart hits the dartboard"}, \\ N &= \text{"Nick's dart hits the dartboard"}. \end{aligned}$$

Use the formal definition of conditional probability to determine

$$\Pr(E \mid E \cup N)$$

and

$$\Pr(E \cap N \mid E \cup N).$$

Show your work.

Solution: We start by computing a probability that we will need later. We know, from inclusion-exclusion, that

$$\Pr(E \cup N) = \Pr(E) + \Pr(N) - \Pr(E \cap N).$$

Since E and N are independent, we have

$$\Pr(E \cap N) = \Pr(E) \cdot \Pr(N).$$

It follows that

$$\begin{aligned} \Pr(E \cup N) &= \Pr(E) + \Pr(N) - \Pr(E) \cdot \Pr(N) \\ &= p + q - pq. \end{aligned}$$

Using the definition of conditional probability, we get

$$\Pr(E \mid E \cup N) = \frac{\Pr(E \cap (E \cup N))}{\Pr(E \cup N)}.$$

By drawing a Venn diagram, you will see that

$$E \cap (E \cup N) = E.$$

It follows that

$$\begin{aligned} \Pr(E \mid E \cup N) &= \frac{\Pr(E)}{\Pr(E \cup N)} \\ &= \frac{p}{p + q - pq}. \end{aligned}$$

Using the definition of conditional probability, we get

$$\Pr(E \cap N \mid E \cup N) = \frac{\Pr((E \cap N) \cap (E \cup N))}{\Pr(E \cup N)}.$$

By drawing a Venn diagram, you will see that

$$(E \cap N) \cap (E \cup N) = E \cap N.$$

It follows that

$$\begin{aligned} \Pr(E \cap N \mid E \cup N) &= \frac{\Pr(E \cap N)}{\Pr(E \cup N)} \\ &= \frac{\Pr(E) \cdot \Pr(N)}{\Pr(E \cup N)} \\ &= \frac{pq}{p + q - pq}. \end{aligned}$$

Question 4: Let $n \geq 4$ be an integer. Consider a uniformly random permutation of $\{1, 2, \ldots, n\}$ and define the events

A = "1 and 2 are next to each other, with 1 to the left of 2, or 4 and 3 are next to each other, with 4 to the left of 3"

and

B = "1 and 2 are next to each other, with 1 to the left of 2, or 2 and 3 are next to each other, with 2 to the left of 3".

- Determine $\Pr(A)$ and $\Pr(B)$.

  Before you answer this question, spend a few seconds on guessing which probability is larger.

Solution: We start with $\Pr(A)$. Let X be the number of permutations that satisfy the condition for the event A. Then

$$\Pr(A) = \frac{X}{n!}.$$

Thus, it remains to determine X.

- We first determine the number of permutations in which 1 is the left neighbor of 2. Imagine these two digits to be one symbol, say, x. Then we have a new alphabet $\{x, 3, 4, \ldots, n\}$ consisting of $n - 1$ symbols. This set has $(n - 1)!$ many permutations.
- By the same reasoning, there are $(n - 1)!$ many permutations in which 4 is the left neighbor of 3.
- We next determine the number of permutations in which 1 is the left neighbor of 2 and 4 is the left neighbor of 3. We imagine the two digits 1 and 2 to be one symbol, say, x, and the two digits 4 and 3 to be one symbol, say y. Then we have a new alphabet $\{x, y, 5, 6, \ldots, n\}$ consisting of $n - 2$ symbols. This set has $(n - 2)!$ many permutations.

Using inclusion-exclusion, we conclude that

$$X = (n-1)! + (n-1)! - (n-2)! = (2n-3) \cdot (n-2)!.$$

This gives

$$\Pr(A) = \frac{X}{n!} = \frac{(2n-3) \cdot (n-2)!}{n!} = \frac{2n-3}{n(n-1)}.$$

Next we determine $\Pr(B)$. Let Y be the number of permutations that satisfy the condition for the event B. Then

$$\Pr(B) = \frac{Y}{n!}.$$

Thus, it remains to determine Y.

- We have seen above that the number of permutations in which 1 is the left neighbor of 2 is equal to $(n-1)!$.
- By the same reasoning, there are $(n-1)!$ many permutations in which 2 is the left neighbor of 3.
- We next determine the number of permutations in which 1 is the left neighbor of 2 and 2 is the left neighbor of 3. We imagine the three digits 1, 2, and 3 to be one symbol, say, z. Then we have a new alphabet $\{z, 4, 5, \ldots, n\}$ consisting of $n-2$ symbols. This set has $(n-2)!$ many permutations.

Using inclusion-exclusion, we conclude that

$$Y = (n-1)! + (n-1)! - (n-2)! = (2n-3) \cdot (n-2)!.$$

This gives

$$\Pr(B) = \frac{Y}{n!} = \frac{(2n-3) \cdot (n-2)!}{n!} = \frac{2n-3}{n(n-1)}.$$

Note that $\Pr(A) = \Pr(B)$.

Question 5: Let A be an event in some probability space $(S, \Pr)$. You are given that the events A and A are independent[1]. Determine $\Pr(A)$. Show your work.

Solution: Since A and A are independent, we have

$$\Pr(A \cap A) = \Pr(A) \cdot \Pr(A).$$

Since $A \cap A = A$, we have

$$\Pr(A \cap A) = \Pr(A).$$

[1] This is not a typo.

By combining these equations, we get

$$\Pr(A) = \Pr(A) \cdot \Pr(A).$$

Let $p = \Pr(A)$. Then we have

$$p = p^2,$$

implying that $p = 0$ or $p = 1$.

Remark: If $p = 0$, then $A = \emptyset$. It is true that $\emptyset$ and $\emptyset$ are two independent events. If $p = 1$, then $A = S$. It is true that S and S are two independent events.

Question 6: Three people P_1, P_2, and P_3 are in a dark room. Each person has a bag containing one red hat and one blue hat. Each person chooses a uniformly random hat from her bag and puts it on her head. Afterwards, the lights are turned on.

Each person does not know the color of her hat, but can see the colors of the other two hats. Each person P_i can do one of the following:

- Person P_i announces "my hat is red".
- Person P_i announces "my hat is blue".
- Person P_i says "I pass".

The game is a *success* if at least one person announces the correct color of her hat and no person announces the wrong color of her hat. (If a person passes, then she does not announce any color.)

- Assume person P_1 announces "my hat is red" and both P_2 and P_3 pass. Define the event

  $$A = \text{"the game is a success."}$$

  Determine $\Pr(A)$. Show your work.

- Assume each person P_i does the following:
  - If the two hats that P_i sees have different colors, then P_i passes.
  - If the two hats that P_i sees are both red, then P_i announces "my hat is blue".
  - If the two hats that P_i sees are both blue, then P_i announces "my hat is red".

  Define the event

  $$B = \text{"the game is a success."}$$

  Determine $\Pr(B)$. Show your work.

Solution: We start with $\Pr(A)$. The game is a success if and only if P_1 announces the correct color of her hat. Since the color of P_1's hat is random (red with probability 1/2 and blue with probability 1/2), it follows that

$$\Pr(A) = 1/2.$$

Next we determine $\Pr(B)$. The sample space S is the possible color-sequences for the three hats:

$$S = \{rrr, rrb, rbr, brr, rbb, brb, bbr, bbb\};$$

the first letter indicates the color of P_1's hat, the second letter indicates the color of P_2's hat, and the third letter indicates the color of P_3's hat.

Note that each color-sequence has a probability of 1/8.

- Assume the color-sequence is rrr. Then each person sees two red hats. According to the rules of the game, each person announces "my hat is blue". Thus, the game is a failure.

- Assume the color-sequence is bbb. Then each person sees two blue hats. According to the rules of the game, each person announces "my hat is red". Thus, the game is a failure.

- Assume the color-sequence has two reds and one blue.

  - Each person with a red hat sees one red and one blue hat. According to the rules, this person passes.
  - The person with a blue hat sees two red hats. According to the rules of the game, this person announces "my hat is blue".

  Thus, the game is a success.

- Assume the color-sequence has one red and two blues. (Of course, this case is symmetric to the previous case.)

  - Each person with a blue hat sees one red and one blue hat. According to the rules, this person passes.
  - The person with a red hat sees two blue hats. According to the rules of the game, this person announces "my hat is red".

  Thus, the game is a success.

To summarize, for 2 elements in the sample space, the game is a failure. For the remaining 6 elements, the game is a success. Therefore,

$$\Pr(B) = 6/8 = 3/4.$$

Note that this second strategy is better than the first one.

Question 7: Let $n \geq 0$ be an integer. In this question, you will prove that

$$\sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k} = \frac{1}{n+1} \left(2^{n+1} - 1 \right). \tag{3}$$

There are $n+1$ students in Carleton's Computer Science program. We denote these students by $P_1, P_2, \ldots, P_{n+1}$. We play the following game:

1. We choose a uniformly random subset X of $\{P_1, P_2, \ldots, P_{n+1}\}$.
2. (a) If $X \neq \emptyset$, then we choose a uniformly random student in X. The chosen student wins a six-pack of cider.
   (b) If $X = \emptyset$, then nobody wins the six-pack.

The random choices made are independent of each other.

- Define the event
  $$A_0 = \text{"nobody wins the six-pack"}.$$
  Determine $\Pr(A_0)$. Justify your answer.

  Solution: The event A_0 occurs if and only if the set X is empty. Since X is chosen uniformly at random from all 2^{n+1} subsets of $\{P_1, P_2, \ldots, P_{n+1}\}$, it follows that
  $$\Pr(A_0) = \Pr(X = \emptyset) = 1/2^{n+1}.$$

- For each $i = 1, 2, \ldots, n+1$, define the event
  $$A_i = \text{"student } P_i \text{ wins the six-pack"}.$$
  Explain in plain English, and in at most two sentences, why
  $$\Pr(A_1) = \Pr(A_2) = \ldots = \Pr(A_{n+1}).$$

  Solution: By symmetry, no student has an advantage over the other students. Therefore, each student has the same probability of winning the six-pack.

- Prove that
  $$\Pr(A_1) = \frac{1 - 1/2^{n+1}}{n+1}.$$

  Solution: Since exactly one of the events $A_0, A_1, \ldots, A_{n+1}$ is guaranteed to occur, we have
  $$\Pr(A_0) + \Pr(A_1) + \cdots + \Pr(A_{n+1}) = 1.$$
  Let $p = \Pr(A_1)$. Then we get
  $$1/2^{n+1} + (n+1)p = 1.$$
  Solving for p gives us
  $$\Pr(A_1) = p = \frac{1 - 1/2^{n+1}}{n+1}.$$

- For each k with $0 \leq k \leq n$, define the event

$$B_k = \text{“}X \text{ has size } k+1 \text{ and } P_1 \text{ wins the six-pack”}.$$

Prove that

$$\Pr(B_k) = \frac{\binom{n}{k}}{2^{n+1}} \cdot \frac{1}{k+1}.$$

Solution: The event B_k occurs if and only if both of the following two conditions are satisfied:

  - $|X| = k+1$ and $P_1 \in X$.
  - P_1 is chosen from X.

How many $(k+1)$-element subsets are there that contain P_1? This is the same as counting the k-element subsets of $\{P_2, P_3, \ldots, P_{n+1}\}$. Thus, the answer is $\binom{n}{k}$.

If the subset X has size $k+1$ and contains P_1, then the probability that P_1 is chosen is equal to $1/(k+1)$. It follows that

$$\Pr(B_k) = \frac{\binom{n}{k}}{2^{n+1}} \cdot \frac{1}{k+1}.$$

If you have your doubts about this derivation, let us do this more carefully (you may read Section 5.9.3 in the textbook): We define the events

$$C = \text{“}|X| = k+1 \text{ and } P_1 \in X\text{”}$$

and

$$D = \text{“}P_1 \text{ is chosen from } X\text{”}.$$

Then the event B_k is equivalent to the event $C \cap D$. Thus,

$$\begin{aligned} \Pr(B_k) &= \Pr(C \cap D) \\ &= \Pr(C) \cdot \Pr(D \mid C). \end{aligned}$$

Since the event C occurs for $\binom{n}{k}$ many subsets X, we have

$$\Pr(C) = \frac{\binom{n}{k}}{2^{n+1}}.$$

To determine $\Pr(D \mid C)$, we assume that the event C occurs. Thus, X has size $k+1$ and it contains P_1. Then the event D occurs if and only if P_1 is chosen. Since the algorithm chooses a uniformly random element in X, it follows that

$$\Pr(D \mid C) = \frac{1}{k+1}.$$

We conclude that

$$\Pr(B_k) = \frac{\binom{n}{k}}{2^{n+1}} \cdot \frac{1}{k+1}.$$

- Express the event A_1 in terms of the events $B_0, B_1, \ldots, B_n$.

  Solution: The event A_1 occurs if and only if P_1 wins the six-pack. This can only happen if the subset X is non-empty. Since the size of X can be any of the numbers $1, 2, \ldots, n+1$, it follows that

  $$A_1 \text{ if and only if } B_0 \vee B_1 \vee \cdots \vee B_n.$$

- Prove that (3) holds by combining the results of the previous parts.

  Solution: Note that the events $B_0, B_1, \ldots, B_n$ are pairwise disjoint. From the previous parts of this question, we get

  $$\begin{aligned} \Pr(A_1) &= \Pr(B_0 \vee B_1 \vee \cdots \vee B_n) \\ &= \sum_{k=0}^{n} \Pr(B_k) \\ &= \sum_{k=0}^{n} \frac{\binom{n}{k}}{2^{n+1}} \cdot \frac{1}{k+1}. \end{aligned}$$

  We have obtained two expressions for $\Pr(A_1)$. These two expressions must be equal. If we multiply both by 2^{n+1}, then we get (3).

Question 8: You roll a fair die once. Define the events

$$\begin{aligned} A &= \text{“the result is even”}, \\ B &= \text{“the result is odd”}, \\ C &= \text{“the result is at most 4”}. \end{aligned}$$

For each of the following questions, justify your answer.

- Are the events A and B independent?
- Are the events A and C independent?
- Are the events B and C independent?

Solution: The sample space is $S = \{1, 2, 3, 4, 5, 6\}$. As subsets of the sample space, the events are

$$A = \{2, 4, 6\},$$

$$B = \{1, 3, 5\},$$

$$C = \{1, 2, 3, 4\}.$$

We have

$$\Pr(A) = |A|/|S| = 3/6 = 1/2,$$
$$\Pr(B) = |B|/|S| = 3/6 = 1/2,$$
$$\Pr(C) = |C|/|S| = 4/6 = 2/3.$$

Since $A \cap B = \emptyset$, we have

$$\Pr(A \cap B) = \Pr(\emptyset) = 0.$$

Thus,

$$\Pr(A \cap B) \neq \Pr(A) \cdot \Pr(B),$$

i.e., A and B are not independent.

We have

$$\Pr(A \cap C) = \Pr(\{2, 4\}) = 2/6 = 1/3,$$

implying that

$$\Pr(A \cap C) = \Pr(A) \cdot \Pr(C),$$

i.e., A and C are independent.

We have

$$\Pr(B \cap C) = \Pr(\{1, 3\}) = 2/6 = 1/3,$$

implying that

$$\Pr(B \cap C) = \Pr(B) \cdot \Pr(C),$$

i.e., B and C are independent.

Question 9: Let n be a large power of two (thus, $\log n$ is an integer). Consider a binary string $s = s_1 s_2 \ldots s_n$, where each bit s_i is 0 with probability $1/2$, and 1 with probability $1/2$, independently of the other bits.

A *run of length* k is a substring of length k, all of whose bits are equal. In class, we have seen that it is very likely that the bitstring s contains a run of length at least $\log n - 2 \log\log n$. In this exercise, you will prove that it is very unlikely that s contains a run of length more than $2 \log n$.

- Let k be an integer with $1 \leq k \leq n$. Define the event

  $$A = \text{"the bitstring } s \text{ contains a run of length at least } k\text{"}.$$

  For each i with $1 \leq i \leq n - k + 1$, define the event

  $$A_i = \text{"the substring } s_i s_{i+1} \ldots s_{i+k-1} \text{ is a run"}.$$

  Use the Union Bound (Lemma 5.3.5 on page 135 of the textbook) to prove that

  $$\Pr(A) \leq \frac{n - k + 1}{2^{k-1}}.$$

- Let $k = 2 \log n$. Prove that

$$\Pr(A) \leq 2/n.$$

Solution: First, we express the event A in terms of the events A_i: A run of length at least k can start at any of the positions $1, 2, \ldots, n-k+1$, implying that

$$A \text{ if and only if } A_1 \vee A_2 \vee \cdots \vee A_{n-k+1}.$$

Thus, we have

$$\Pr(A) = \Pr\left(A_1 \vee A_2 \vee \cdots \vee A_{n-k+1}\right).$$

Note that the events $A_1, \ldots, A_{n-k+1}$ are *not* pairwise disjoint. By applying the Union-Bound to the right-hand side, we get

$$\begin{aligned} \Pr(A) &\leq \Pr(A_1) + \Pr(A_2) + \cdots + \Pr(A_{n-k+1}) \\ &= \sum_{i=1}^{n-k+1} \Pr(A_i). \end{aligned}$$

The event A_i occurs if and only the substring $s_i \ldots s_{i+k-1}$ consists of k zeroes or k ones. Since the bits are generated uniformly at random, we get

$$\begin{aligned} \Pr(A_i) &= \Pr(k \text{ zeroes}) + \Pr(k \text{ ones}) \\ &= (1/2)^k + (1/2)^k \\ &= 1/2^{k-1}. \end{aligned}$$

We conclude that

$$\begin{aligned} \Pr(A) &\leq \sum_{i=1}^{n-k+1} \Pr(A_i) \\ &= \sum_{i=1}^{n-k+1} 1/2^{k-1} \\ &= \Pr(A) \leq \frac{n-k+1}{2^{k-1}}. \end{aligned}$$

Now we take $k = 2 \log n$. Since

$$2^k = 2^{2 \log n} = 2^{\log(n^2)} = n^2,$$

we get

$$\Pr(A) \leq \frac{n-k+1}{2^{k-1}}$$

$$
\begin{aligned}
&\leq \frac{n}{2^{k-1}} \\
&= \frac{2n}{2^k} \\
&= \frac{2n}{n^2} \\
&= \frac{2}{n}.
\end{aligned}
$$