COMP 2804 — Solutions Assignment 4

Question 1: On the first page of your assignment, write your name and student number.

Solution:

- Name: Lionel Messi

- Student number: 10

Question 2: Two players P_1 and P_2 play a game in which they take turns flipping, independently, a fair coin: First P_1 flips the coin, then P_2 flips the coin, then P_1 flips the coin, then P_2 flips the coin, etc. The game ends as soon as the sequence of coin flips contains either HH or TT. The player who flips the coin for the last time is the winner of the game. For example, if the sequence of coin flips is $HTHTHH$, then P_2 wins the game.

Determine the probability that P_1 wins the game. Show your work.

Solution: Each possible sequence of coin flips has one of the following four forms (to convince yourself that this is correct, "reconstruct" these sequences backwards):

1. $(HT)^n HH$ for some $n \geq 0$. This means: n times HT, followed by HH. Since these sequences have an even length, P_2 wins.

2. $T(HT)^n HH$ for some $n \geq 0$. This means: T, followed by n times HT, followed by HH. Since these sequences have an odd length, P_1 wins.

3. $(TH)^n TT$ for some $n \geq 0$. Since these sequences have an even length, P_2 wins.

4. $H(TH)^n TT$ for some $n \geq 0$. Since these sequences have an odd length, P_1 wins.

If we define the event

$$A = \text{``}P_1 \text{ wins''},$$

then we have

$$
\begin{aligned}
\Pr(A) &= \Pr\left(\, T(HT)^n HH \text{ or } H(TH)^n TT \text{ for some } n \geq 0 \,\right) \\
&= \sum_{n=0}^{\infty} \Pr\left(T(HT)^n HH\right) + \sum_{n=0}^{\infty} \Pr\left(H(TH)^n TT\right).
\end{aligned}
$$

Since the coin is fair, both sums are equal and we get

$$
\Pr(A) = 2\sum_{n=0}^{\infty} \Pr\left(T(HT)^n HH\right) \qquad \text{(there are } 2n+3 \text{ coin flips)}
$$

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$$= 2\sum_{n=0}^{\infty} (1/2)^{2n+3}$$

$$= (1/4)\sum_{n=0}^{\infty} (1/2)^{2n}$$

$$= (1/4)\sum_{n=0}^{\infty} (1/4)^{n}.$$

In class, we have seen that, for $0 < x < 1$,

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

By taking $x = 1/4$, we get

$$\begin{aligned} \Pr(A) &= \frac{1}{4} \cdot \frac{1}{1-1/4} \\ &= \frac{1}{4} \cdot \frac{1}{3/4} \\ &= \frac{1}{3}. \end{aligned}$$

Question 3: Assume we flip a fair coin twice, independently of each other. Define the following random variables:

$$\begin{aligned} X &= \text{the number of heads,} \\ Y &= \text{the number of tails,} \\ Z &= \text{the number of heads times the number of tails.} \end{aligned}$$

- Determine the expected values of these three random variables. Show your work.

- Are X and Y independent random variables? Justify your answer.

- Are X and Z independent random variables? Justify your answer.

- Are Y and Z independent random variables? Justify your answer.

Solution: In class, we have seen the following: Assume we have a coin that comes up heads with probability p and comes up tails with probability $1 - p$. If we flip this coin n times (independently), then the expected number of heads is pn.

In this question, $p = 1/2$ and $n = 2$. Therefore,

$$\mathbb{E}(X) = pn = 1$$

and

$$\mathbb{E}(Y) = pn = 1.$$

Alternatively, we can compute $\mathbb{E}(X)$ "by hand": What are the possible values for X:

1. $X = 0$: This happens if we flip the sequence TT. This happens with probability $1/4$.

2. $X = 1$: This happens if we flip the sequence HT or the sequence TH. This happens with probability $1/2$.

3. $X = 2$: This happens if we flip the sequence HH. This happens with probability $1/4$.

This gives

$$
\begin{aligned}
\mathbb{E}(X) &= 0 \cdot \Pr(X = 0) + 1 \cdot \Pr(X = 1) + 2 \cdot \Pr(X = 2) \\
&= 0 + 1 \cdot 1/2 + 2 \cdot 1/4 \\
&= 1.
\end{aligned}
$$

In yet another way, we can use indicator random variables to determine $\mathbb{E}(X)$; we did this in class, so you can find this in your notes and in the textbook.

By symmetry, we have $\mathbb{E}(Y) = 1$. Another way to get this: Since $X + Y = 2$, we get

$$
\mathbb{E}(Y) = \mathbb{E}(2 - X) = 2 - \mathbb{E}(X) = 2 - 1 = 1.
$$

Now we consider the random variable Z. Note that $Z = X \cdot Y$. What are the possible values for Z:

1. $Z = 0$: This happens if we flip the sequence HH or the sequence TT. This happens with probability $1/2$.

2. $Z = 1$: This happens if we flip the sequence HT or the sequence TH. This happens with probability $1/2$.

Thus, we get

$$
\begin{aligned}
\mathbb{E}(Z) &= 0 \cdot \Pr(Z = 0) + 1 \cdot \Pr(Z = 1) \\
&= 0 + 1 \cdot 1/2 \\
&= 1/2.
\end{aligned}
$$

Remark: This shows that

$$
\mathbb{E}(X \cdot Y) \neq \mathbb{E}(X) \cdot \mathbb{E}(Y).
$$

Are X and Y independent random variables? Since $X + Y = 2$, the following holds: If $X = 2$, then Y must be 0; in other words, Y cannot be 2. Based on this, we have (for example),

$$
\Pr(X = 2 \text{ and } Y = 2) = 0,
$$

whereas

$$
\Pr(X = 2) \cdot \Pr(Y = 2) = 1/4 \cdot 1/4 \neq 0.
$$

We conclude that X and Y are not independent.

Are X and Z independent random variables? Since $Z = XY$, the following holds: If $X = 0$, then Z must be 0; in other words, Z cannot be 1. Based on this, we have (for example),

$$\Pr(X = 0 \text{ and } Z = 1) = 0,$$

whereas

$$\Pr(X = 0) \cdot \Pr(Z = 1) = 1/4 \cdot 1/2 \neq 0.$$

We conclude that X and Z are not independent.

Are Y and Z independent random variables? Since $Z = XY$, the following holds: If $Y = 0$, then Z must be 0; in other words, Z cannot be 1. Based on this, we have (for example),

$$\Pr(Y = 0 \text{ and } Z = 1) = 0,$$

whereas

$$\Pr(Y = 0) \cdot \Pr(Z = 1) = 1/4 \cdot 1/2 \neq 0.$$

We conclude that Y and Z are not independent.

Question 4: Consider the sample space $S = \{1, 2, 3 \ldots, 10\}$. We choose a uniformly random element x in S. Define the following random variables:

$$X = \begin{cases} 0 & \text{if } x \in \{1, 2\}, \\ 1 & \text{if } x \in \{3, 4, 5, 6\}, \\ 2 & \text{if } x \in \{7, 8, 9, 10\} \end{cases}$$

and

$$Y = \begin{cases} 0 & \text{if } x \text{ is even}, \\ 1 & \text{if } x \text{ is odd}. \end{cases}$$

Are X and Y independent random variables? Justify your answer.

Solution: We are going to show that X and Y are independent. Since X has 3 possible values and Y has 2 possible values, we have to verify $3 \cdot 2 = 6$ equations:

1. $X = 0$ and $Y = 0$:
$$\Pr(X = 0 \wedge Y = 0) = \Pr(\{2\}) = 1/10.$$
$$\Pr(X = 0) \cdot \Pr(Y = 0) = 2/10 \cdot 1/2 = 1/10.$$

2. $X = 0$ and $Y = 1$:
$$\Pr(X = 0 \wedge Y = 1) = \Pr(\{1\}) = 1/10.$$
$$\Pr(X = 0) \cdot \Pr(Y = 1) = 2/10 \cdot 1/2 = 1/10.$$

3. $X = 1$ and $Y = 0$:
$$\Pr(X = 1 \wedge Y = 0) = \Pr(\{4, 6\}) = 2/10 = 1/5.$$
$$\Pr(X = 1) \cdot \Pr(Y = 0) = 4/10 \cdot 1/2 = 1/5.$$

4. $X = 1$ and $Y = 1$:

$$\Pr(X = 1 \wedge Y = 1) = \Pr(\{3, 5\}) = 2/10 = 1/5.$$

$$\Pr(X = 1) \cdot \Pr(Y = 1) = 4/10 \cdot 1/2 = 1/5.$$

5. $X = 2$ and $Y = 0$:

$$\Pr(X = 2 \wedge Y = 0) = \Pr(\{8, 10\}) = 2/10 = 1/5.$$

$$\Pr(X = 2) \cdot \Pr(Y = 0) = 4/10 \cdot 1/2 = 1/5.$$

6. $X = 2$ and $Y = 1$:

$$\Pr(X = 2 \wedge Y = 1) = \Pr(\{7, 9\}) = 2/10 = 1/5.$$

$$\Pr(X = 2) \cdot \Pr(Y = 1) = 4/10 \cdot 1/2 = 1/5.$$

Question 5: In order to attract more customers, the Hyacintho Cactus Bar and Grill in downtown Ottawa organizes a game night, hosted by their star employee Tan Tran.

After paying \$26, a player gets two questions P and Q. If the player gives the correct answer to question P, this player wins \$30; if the player gives the correct answer to question Q, this player wins \$60. A player can choose between the following two options:

1. Start with question P. In this case, the player is allowed to answer question Q only if the answer to question P is correct.

2. Start with question Q. In this case, the player is allowed to answer question P only if the answer to question Q is correct.

Elisa decides to play this game. The probability that Elisa correctly answers question P is equal to $1/2$, whereas she correctly answers question Q with probability $1/3$. The events of correctly answering are independent.

- Assume Elisa chooses the first option. Define the random variable X to be the amount of money that Elisa wins (this includes the \$26 that she has to pay in order to play the game). Determine the expected value $\mathbb{E}(X)$. Show your work.

- Assume Elisa chooses the second option. Define the random variable Y to be the amount of money that Elisa wins (this includes the \$26 that she has to pay in order to play the game). Determine the expected value $\mathbb{E}(Y)$. Show your work.

Solution: We start with the first option: Elisa starts with question P. We consider all possibilities:

1. The answer to P is wrong. In this case, the game is over, and Elisa has lost \$26, i.e., $X = -26$. This happens with probability $1/2$.

2. The answer to P is correct, whereas the answer to question Q is wrong. In this case, $X = -26 + 30 = 4$. The probability that this happens is equal to

$$
\begin{aligned}
\Pr(P \text{ correct}) \cdot \Pr(Q \text{ wrong} \mid P \text{ correct}) &= \Pr(P \text{ correct}) \cdot \Pr(Q \text{ wrong}) \\
&= 1/2 \cdot 2/3 \\
&= 1/3.
\end{aligned}
$$

3. The answer to P is correct and the answer to question Q is correct. In this case, $X = -26 + 30 + 60 = 64$. The probability that this happens is equal to

$$
\begin{aligned}
\Pr(P \text{ correct}) \cdot \Pr(Q \text{ correct} \mid P \text{ correct}) &= \Pr(P \text{ correct}) \cdot \Pr(Q \text{ correct}) \\
&= 1/2 \cdot 1/3 \\
&= 1/6.
\end{aligned}
$$

Thus, the possible values for X are -26, 4, and 64. We conclude that

$$
\mathbb{E}(X) = (-26) \cdot 1/2 + 4 \cdot 1/3 + 64 \cdot 1/6 = -1.
$$

Next we do the second option: Elisa starts with question Q. We consider all possibilities:

1. The answer to Q is wrong. In this case, the game is over, and Elisa has lost \$26, i.e., $X = -26$. This happens with probability $2/3$.

2. The answer to Q is correct, whereas the answer to question P is wrong. In this case, $X = -26 + 60 = 34$. The probability that this happens is equal to

$$
\begin{aligned}
\Pr(Q \text{ correct}) \cdot \Pr(P \text{ wrong} \mid Q \text{ correct}) &= \Pr(Q \text{ correct}) \cdot \Pr(P \text{ wrong}) \\
&= 1/3 \cdot 1/2 \\
&= 1/6.
\end{aligned}
$$

3. The answer to Q is correct and the answer to question P is correct. In this case, $X = -26 + 60 + 30 = 64$. The probability that this happens is equal to

$$
\begin{aligned}
\Pr(Q \text{ correct}) \cdot \Pr(P \text{ correct} \mid Q \text{ correct}) &= \Pr(Q \text{ correct}) \cdot \Pr(P \text{ correct}) \\
&= 1/3 \cdot 1/2 \\
&= 1/6.
\end{aligned}
$$

Thus, the possible values for X are -26, 34, and 64. We conclude that

$$
\mathbb{E}(X) = (-26) \cdot 2/3 + 34 \cdot 1/6 + 64 \cdot 1/6 = -1.
$$

Question 6: Consider a uniformly random permutation $a_1, a_2, \ldots, a_n$ of the set $\{1, 2, \ldots, n\}$. Define the random variable X to be the number of ordered pairs (i, j) with $1 \leq i < j \leq n$ for which $a_i = j$ and $a_j = i$. Determine the expected value $\mathbb{E}(X)$ of X. Show your work.

  Hint: Use indicator random variables.

Solution: For each ordered pair (i, j) with $1 \leq i < j \leq n$, we define the indicator random variable

$$X_{ij} = \begin{cases} 1 & \text{if } a_i = j \text{ and } a_j = i, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$X = \sum_{i,j} X_{ij}$$

and

$$\begin{aligned} \mathbb{E}(X) &= \mathbb{E}\left(\sum_{i,j} X_{ij}\right) \\ &= \sum_{i,j} \mathbb{E}(X_{ij}) \\ &= \sum_{i,j} \Pr(X_{ij} = 1). \end{aligned}$$

Consider two fixed indices i and j with $1 \leq i < j \leq n$. Let N be the number of permutations $a_1, a_2, \ldots, a_n$ of the set $\{1, 2, \ldots, n\}$ for which $a_i = j$ and $a_j = i$. Then

$$\Pr(X_{ij} = 1) = \frac{N}{n!}.$$

What is the value of N: In each permutation that is counted in N, the value j must be at position i and the value i must be at position j. So we are really counting the permutations of the set

$$\{1, 2, \ldots, i-1, i+1, i+2, \ldots, j-1, j+1, j+2, \ldots, n\}.$$

Since this set has $n - 2$ elements, we have $N = (n-2)!$, implying that

$$\Pr(X_{ij} = 1) = \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}.$$

We conclude that

$$\begin{aligned} \mathbb{E}(X) &= \sum_{i,j} \Pr(X_{ij} = 1) \\ &= \sum_{i,j} \frac{1}{n(n-1)} \end{aligned}$$

$$= \binom{n}{2} \cdot \frac{1}{n(n-1)}$$
$$= \frac{n(n-1)}{2} \cdot \frac{1}{n(n-1)}$$
$$= 1/2.$$

Question 7: Nick[1] wants to know how many students cheat on the assignments. One approach is to ask every student "Did you cheat?". This obviously does not work, because every student will answer "I did not cheat". Instead, Nick uses the following ingenious scheme, which gives a reasonable estimate of the number of cheaters, without identifying them.

We denote the students by $S_1, S_2, \ldots, S_n$. Let k denote the number of cheaters. Nick knows the value of n, but he does not know the value of k.

For each i with $1 \leq i \leq n$, Nick does the following:

1. Nick meets student S_i and asks "Did you cheat?".

2. Student S_i flips a fair coin twice, independently of each other; S_i does not show the results of the coin flips to Nick.

   (a) If the coin flips are HH or HT, then S_i is honest in answering the question: If S_i is a cheater, then he answers "I cheated"; otherwise, he answers "I did not cheat".

   (b) If the coin flips are TH, then S_i answers "I cheated".

   (c) If the coin flips are TT, then S_i answers "I did not cheat".

- Define the random variable X to be the number of students who answer "I cheated". Determine the expected value $\mathbb{E}(X)$ of X.

  Hint: For each i, use an indicator random variable X_i which indicates whether or not S_i answers "I cheated". If S_i is a cheater, what is $\mathbb{E}(X_i)$? If S_i is not a cheater, what is $\mathbb{E}(X_i)$?

- Define the random variable
  $$Y = 2X - n/2.$$

  Prove that $\mathbb{E}(Y) = k$. In words, the expected value of Y is equal to the number of cheaters.

Solution: For each $i = 1, 2, \ldots, n$, we define the indicator random variable

$$X_i = \begin{cases} 1 & \text{if student } S_i \text{ answers "I cheated"}, \\ 0 & \text{otherwise}. \end{cases}$$

---

[1]your friendly TA

Then
$$X = \sum_{i=1}^{n} X_i$$

and

$$
\begin{aligned}
\mathbb{E}(X) &= \mathbb{E}\left(\sum_{i=1}^{n} X_i\right) \\
&= \sum_{i=1}^{n} \mathbb{E}\left(X_i\right) \\
&= \sum_{i=1}^{n} \Pr\left(X_i = 1\right) \\
&= \sum_{i=1}^{n} \Pr\left(S_i \text{ answers "I cheated"}\right).
\end{aligned}
$$

1. Assume S_i is a cheater. Then S_i answers "I cheated" if the coin flips are HH or HT or TH. Thus,
$$\Pr\left(S_i \text{ answers "I cheated"}\right) = 3/4.$$

2. Assume S_i is not a cheater. Then S_i answers "I cheated" if the coin flips are TH. Thus,
$$\Pr\left(S_i \text{ answers "I cheated"}\right) = 1/4.$$

We have just seen that

$$\mathbb{E}(X) = \sum_{i=1}^{n} \Pr\left(S_i \text{ answers "I cheated"}\right).$$

In this sum, k terms are equal to $3/4$ (because there are k cheaters), whereas the other $n-k$ terms are equal to $1/4$ (because there are $n-k$ non-cheaters). We conclude that

$$\mathbb{E}(X) = 3/4 \cdot k + 1/4 \cdot (n-k) = n/4 + k/2.$$

Using the Linearity of expectation, we get

$$
\begin{aligned}
\mathbb{E}(Y) &= \mathbb{E}(2X - n/2) \\
&= 2 \cdot \mathbb{E}(X) - n/2 \\
&= 2 \cdot (n/4 + k/2) - n/2 \\
&= k.
\end{aligned}
$$

Remark: By determining the value of the random variable Y, Nick will get a reasonable estimate of the number of cheaters. Note that each student gives an honest answer with probability $1/2$, and a random answer with probability $1/2$. Because of this, Nick does not know who the cheaters are.