COMP 2804 — Assignment 1

Due: Thursday February 1, before 11:59pm.

Assignment Policy:

- Your assignment must be submitted as one single PDF file through cuLearn.
- Late assignments will not be accepted.
- You are encouraged to collaborate on assignments, but at the level of discussion only. When writing your solutions, you must do so in your own words.
- Past experience has shown conclusively that those who do not put adequate effort into the assignments do not learn the material and have a probability near 1 of doing poorly on the exams.
- When writing your solutions, you must follow the guidelines below.
  - You must justify your answers.
  - The answers should be concise, clear and neat.
  - When presenting proofs, every step should be justified.

Question 1:

- Write your name and student number.

Question 2: Let S be the set of all integers $x > 6543$ such that the decimal representation of x has distinct digits, none of which is equal to 7, 8, or 9. (The decimal representation does not have leading zeros.) Determine the size of the set S.

(You do not get marks if you write out all elements of S.)

Question 3: Let S be the set of all integers $x \in \{1, 2, \ldots, 100\}$ such that the decimal representation of x does not contain the digit 4. (The decimal representation does not have leading zeros.)

- Determine the size of the set S without using the Complement Rule.
- Use the Complement Rule to determine the size of the set S.

(You do not get marks if you write out all numbers from 1 to 100 and mark those that belong to the set S.)

Question 4: The Ottawa Senators and the Toronto Maple Leafs play a best-of-7 series: These two hockey teams play games against each other, and the first team to win 4 games wins the series. Each game has a winner (thus, no game ends in a tie).

A sequence of games can be described by a string consisting of the characters S (indicating that the Senators win the game) and L (indicating that the Leafs win the game). Two possible ways for the Senators to win the series are (L, S, S, S, S) and (S, L, S, L, S, S).

Determine the number of ways in which the Senators can win the series.

(You do not get marks if you write out all possible ways.)

Question 5: Let $m \geq 2$ and $n \geq 2$ be even integers. You are given m beer bottles $B_1, B_2, \ldots, B_m$ and n cider bottles $C_1, C_2, \ldots, C_n$. Assume you arrange these $m + n$ bottles on a horizontal line such that

- the leftmost $m/2$ bottles are all beer bottles, and
- the rightmost $n/2$ bottles are all cider bottles.

How many such arrangements are there? (The order of the bottles matters.)

Question 6: Consider strings consisting of 40 characters, where each character is one of the letters a, b, and c. Such a string is called *cool* if it contains exactly 8 many a's or exactly 7 many b's. Determine the number of cool strings.

Question 7: Consider a group of 100 students. In this group, 13 students like Donald Trump, 25 students like Justin Bieber, and 8 students like Donald Trump and like Justin Bieber. How many students in this group do not like Donald Trump and do not like Justin Bieber?

Question 8: Use Newton's Binomial Theorem to prove that for every integer $n \geq 2$,

$$\sum_{k=0}^{n} \binom{n}{k} (n-1)^{n-k} = n^n. \tag{1}$$

In the rest of this exercise, you will give a combinatorial proof of this identity.

Consider the set $S = \{1, 2, \ldots, n\}$. We have seen in class that the number of functions $f : S \to S$ is equal to n^n.

- Consider a fixed integer k with $0 \leq k \leq n$ and a fixed subset A of S having size k. Determine the number of functions $f : S \to S$ having the property that $f(x) = x$ for all $x \in A$, and $f(x) \neq x$ for all $x \in S \setminus A$.
- Explain why the above part implies the identity in (1).

  Hint: Divide the functions f into groups based on the number of x for which $f(x) = x$.

Question 9: Let $n \geq 1$ be an integer. We consider binary $2 \times n$ matrices, i.e., matrices with 2 rows and n columns, in which each entry is 0 or 1. Any column in such a matrix is of one of four types, based on the bits that occur in this column. We will refer to these types as $\substack{0\\0}$-columns, $\substack{0\\1}$-columns, $\substack{1\\0}$-columns, and $\substack{1\\1}$-columns. For example, in the 2×7 matrix below, the first, second, and fifth columns are $\substack{0\\1}$-columns, the third and seventh columns are $\substack{1\\1}$-columns, the fourth column is a $\substack{0\\0}$-column, and the sixth column is a $\substack{1\\0}$-column.

0	0	1	0	0	1	1
1	1	1	0	1	0	1

For the rest of this exercise, let k be an integer with $0 \leq k \leq 2n$. A binary $2 \times n$ matrix is called *awesome*, if it contains exactly k many 0's.

- How many 1's are there in an awesome $2 \times n$ matrix?
- How many awesome $2 \times n$ matrices are there?
- Let i be an integer and consider an arbitrary awesome $2 \times n$ matrix M with exactly $n - i$ many $\substack{1\\1}$-columns.
  - Prove that $\lceil k/2 \rceil \leq i \leq k$.
  - Determine the number of $\substack{0\\1}$-columns plus the number of $\substack{1\\0}$-columns in M.
- Let i be an integer. Prove that the number of awesome $2 \times n$ matrices with exactly $n - i$ many $\substack{1\\1}$-columns is equal to

$$2^{2i-k} \binom{n}{n-i} \binom{i}{2i-k}.$$

- Use the above results to prove that

$$\sum_{i=\lceil k/2 \rceil}^{k} 2^{2i} \binom{n}{i} \binom{i}{k-i} = 2^k \binom{2n}{k}.$$

Question 10: Let $S_1, S_2, \ldots, S_{50}$ be a sequence consisting of 50 subsets of the set $\{1, 2, \ldots, 55\}$. Assume that each of these 50 subsets consists of at least seven elements.

Use the Pigeonhole Principle to prove that there exist two distinct indices i and j, such that the largest element in S_i is equal to the largest element in S_j.