COMP 2804 — Assignment 2

Due: Thursday February 15, before 11:55pm.

Assignment Policy:

- Your assignment must be submitted as one single PDF file through cuLearn.
- Late assignments will not be accepted.
- You are encouraged to collaborate on assignments, but at the level of discussion only. When writing your solutions, you must do so in your own words.
- Past experience has shown conclusively that those who do not put adequate effort into the assignments do not learn the material and have a probability near 1 of doing poorly on the exams.
- When writing your solutions, you must follow the guidelines below.
  - You must justify your answers.
  - The answers should be concise, clear and neat.
  - When presenting proofs, every step should be justified.

Question 1:

- Write your name and student number.

Question 2: The function $f : \mathbb{N} \to \mathbb{Z}$ is defined by

$$\begin{aligned} f(0) &= 0, \\ f(n) &= f(n-1) + (n^2 - n - 4) \cdot 2^{n-1} \quad \text{if } n \geq 1. \end{aligned}$$

- Determine $f(n)$ for $n = 0, 1, 2, 3, 4, 5$.
- Prove that for every integer $n \geq 0$,

$$f(n) = \left(n^2 - 3n\right) \cdot 2^n.$$

Question 3: You are asked to come up with an exam question about recurrences that is in the same style as Question 2. Thus, you write down some recurrence, which you then solve. Afterwards, you give the recurrence to the students and you give them the solution as well. The students must then prove that the given solution is indeed correct.

This is a painful process, because you must solve the recurrence yourself. Since you are lazy, you start with the following:

Exam Question:

The function $f : \mathbb{N} \to \mathbb{N}$ is defined by

$$\begin{aligned} f(0) &= XXX, \\ f(n) &= f(n-1) + YYY \quad \text{if } n \geq 1. \end{aligned}$$

Prove that for every integer $n \geq 0$,

$$f(n) = 7n^2 - 2n + 9.$$

- Complete the question, i.e., fill in XXX and YYY, so that you obtain a complete recurrence that has the given solution.

Question 4: The sequence of numbers a_n, for $n \geq 0$, is recursively defined as follows:

$$\begin{aligned} a_0 &= 5, \\ a_1 &= 3, \\ a_n &= 6 \cdot a_{n-1} - 9 \cdot a_{n-2} \quad \text{if } n \geq 2. \end{aligned}$$

- Determine a_n for $n = 0, 1, 2, 3, 4, 5$.
- Prove that for every integer $n \geq 0$,

$$a_n = (5 - 4n) \cdot 3^n.$$

Question 5: In this exercise, we consider strings of characters, where each character is an element of $\{a, b, c\}$. For any integer $n \geq 1$, let E_n be the number of such strings of length n that have an even number of c's, and let O_n be the number of such strings of length n that have an odd number of c's. (Recall that 0 is even.)

- Determine E_1, O_1, E_2, and O_2.
- Explain in plain English and at most two sentences why

$$E_n + O_n = 3^n.$$

- Prove that for every integer $n \geq 2$,

$$E_n = 2 \cdot E_{n-1} + O_{n-1}.$$

- Prove that for every integer $n \geq 1$,

$$E_n = \frac{1 + 3^n}{2}.$$

Question 6: A *block* in a bitstring is a maximal consecutive substring of 1's. For example, the bitstring 1100011110100111 has four blocks: 11, 1111, 1, and 111.

For a given integer $n \geq 1$, consider all 2^n bitstrings of length n. Let B_n be the total number of blocks in all these bitstrings.

For example, the left part of the table below contains all 8 bitstrings of length 3. Each entry in the rightmost column shows the number of blocks in the corresponding bitstring. Thus,

$$B_3 = 0 + 1 + 1 + 1 + 1 + 2 + 1 + 1 = 8.$$

0	0	0	0
0	0	1	1
0	1	0	1
1	0	0	1
0	1	1	1
1	0	1	2
1	1	0	1
1	1	1	1

- Determine B_1 and B_2.
- Let $n \geq 3$ be an integer.
  - Consider all bitstrings of length n that start with 0. What is the total number of blocks in these bitstrings?
  - Determine the number of blocks in the bitstring

    $$\underbrace{1 \cdots 1}_{n}.$$

  - Determine the number of blocks in the bitstring

    $$\underbrace{1 \cdots 1}_{n-1} 0.$$

  - Let k be an integer with $2 \leq k \leq n - 1$. Consider all bitstrings of length n that start with

    $$\underbrace{1 \cdots 1}_{k-1} 0.$$

    Prove that the total number of blocks in these bitstrings is equal to

    $$2^{n-k} + B_{n-k}.$$

  - Prove that

    $$B_n = 2 + B_{n-1} + \sum_{k=2}^{n-1} \left(2^{n-k} + B_{n-k} \right).$$

– Use $1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-2} = 2^{n-1} - 1$, to prove that

$$B_n = 2^{n-1} + B_1 + B_2 + \cdots + B_{n-1}. \tag{1}$$

- Prove that (1) also holds for $n = 2$.
- Let $n \geq 3$. Prove that
$$B_n = 2^{n-2} + 2 \cdot B_{n-1}.$$
Hint: Write (1) on one line. Below this line, write (1) with n replaced by $n - 1$.
- Prove that for every $n \geq 1$,
$$B_n = \frac{n+1}{4} \cdot 2^n.$$

Question 7: Let $n \geq 1$ be an integer and consider n beer bottles $B_1, B_2, \ldots, B_n$. In this exercise, we consider different ways to divide these bottles into subsets of size at most 2.

For example, if $n = 6$, then two different ways to do this are

$$\{B_1\}, \{B_2, B_6\}, \{B_3, B_4\}, \{B_5\}$$

and

$$\{B_1, B_4\}, \{B_2\}, \{B_3\}, \{B_5, B_6\}.$$

The order in which we write the subsets does not matter.

For each $n \geq 1$, let W_n be the number of different ways to divide n beer bottles into subsets of size at most 2.

- Determine W_1, W_2, W_3, and W_4.
- Prove that for every integer $n \geq 3$,
$$W_n = W_{n-1} + (n-1) \cdot W_{n-2}.$$

Question 8: Consider the following recursive algorithm, which takes as input a sequence $(a_1, a_2, \ldots, a_n)$ of length n, where $n \geq 1$:

Algorithm MYSTERY$(a_1, a_2, \ldots, a_n)$**:**

    if $n = 1$
    then return the sequence (a_1)
    else $(b_1, b_2, \ldots, b_{n-1}) =$ MYSTERY$(a_1, a_2, \ldots, a_{n-1})$;
        return the sequence $(a_n, b_1, b_2, \ldots, b_{n-1})$
    endif

- Express the output of algorithm MYSTERY$(a_1, a_2, \ldots, a_n)$ in terms of the input sequence $(a_1, a_2, \ldots, a_n)$. Prove that your answer correct.

Question 9: Ever since he was a child, Nick[1] has been dreaming to be like Spiderman. As you all know, Spiderman can climb up the outside of a building; if he is at a particular floor, then, in one step, he can move up several floors. Nick is not that advanced yet. In one step, Nick can move up either one floor or two floors.

Let $n \geq 1$ be an integer and consider a building with n floors, numbered $1, 2, \ldots, n$. (The first floor has number 1; this is not the ground floor.) Nick is standing in front of this building, at the ground level. There are different ways in which Nick can climb to the n-th floor. For example, here are three different ways for the case when $n = 5$:

move up 2 floors, move up 1 floor, move up 2 floors.

move up 1 floor, move up 2 floors, move up 2 floors.

move up 1 floor, move up 2 floors, move up 1 floor, move up 1 floor .

Let S_n be the number of different ways, in which Nick can climb to the n-th floor.

- Determine, S_1, S_2, S_3, and S_4.
- Determine the value of S_n, i.e., express S_n in terms of numbers that we have seen in class. As always, justify your answer.

[1] your friendly TA