# COMP 2804 — Assignment 3

**Due:** Thursday March 22, before 11:55pm, through cuLearn.

**Assignment Policy:**

- Your assignment must be submitted as one single PDF file through cuLearn.
- Late assignments will not be accepted.
- You are encouraged to collaborate on assignments, but at the level of discussion only. When writing your solutions, you must do so in your own words.
- Past experience has shown conclusively that those who do not put adequate effort into the assignments do not learn the material and have a probability near 1 of doing poorly on the exams.
- When writing your solutions, you must follow the guidelines below.
  - You must justify your answers.
  - The answers should be concise, clear and neat.
  - When presenting proofs, every step should be justified.

**Question 1:** Write your name and student number.

**Question 2:** When Tri[1] is a big boy, he wants to have four children. Assuming that the genders of these children are uniformly random, which of the following three events has the highest probability?

1. All four kids are of the same gender.
2. Three kids are of the same gender and the fourth kid is of the opposite gender.
3. Two kids are boys and two kids are girls.

As always, justify your answer.

[1] your friendly TA

**Question 3:** In this exercise, we assume that, when a child is born, its gender and day of birth are uniformly random and independent of other children. Thus, for each $G \in \{\text{boy, girl}\}$ and each

$$D \in \{\text{Sun, Mon, Tue, Wednes, Thurs, Fri, Satur}\},$$

the probability that a child has gender $G$ and is born on a $D$day is equal to $1/14$.

Anil Maheshwari[2] has two children. You are given that at least one of Anil's kids is a boy who was born on a Sunday. Determine the probability that Anil has two boys.

**Question 4:** You are given a fair red die and a fair blue die. Each of these two dice has the letter $a$ on one face, the letter $b$ on two faces, and the letter $c$ on three faces. You roll both dice uniformly at random and independently of each other. Define the events

$$A = \text{“at least one of the two rolls results in the letter } b\text{”}$$

and

$$B = \text{“both rolls result in the same letter”}.$$

- Determine $\Pr(A)$, $\Pr(B)$, and $\Pr(A \mid B)$.

**Question 5:** In a standard deck of 52 cards, each card has a *suit* and a *rank*. There are four suits (spades ♠, hearts ♡, clubs ♣, and diamonds ♢), and 13 ranks (Ace, $2, 3, 4, 5, 6, 7, 8, 9, 10$, Jack, Queen, and King).

A *hand of cards* is a subset consisting of five cards. A hand of cards is called a *straight*, if the ranks of these five cards are consecutive and the cards are not all of the same suit.

An Ace and a 2 are considered to be consecutive, whereas a King and an Ace are also considered to be consecutive. For example, each of the three hands below is a straight:

$$8\spadesuit, 9\heartsuit, 10\diamondsuit, J\spadesuit, Q\clubsuit$$

$$A\diamondsuit, 2\heartsuit, 3\spadesuit, 4\spadesuit, 5\clubsuit$$

$$10\diamondsuit, J\heartsuit, Q\spadesuit, K\spadesuit, A\clubsuit$$

- Assume you get a uniformly random hand of cards. Determine the probability that this hand is a straight.

[2] the guy in the office next to my office

**Question 6:** In this exercise, we consider a standard deck of 52 cards.

- We choose, uniformly at random, one card from the deck. Define the events

$$\begin{aligned} A &= \text{"the rank of the chosen card is Ace"}, \\ B &= \text{"the suit of the chosen card is diamonds"}. \end{aligned}$$

  Are the events $A$ and $B$ independent? As always, justify your answer.

- Assume we remove the Queen of hearts from the deck. We choose, uniformly at random, one card from the remaining 51 cards. Define the events

$$\begin{aligned} C &= \text{"the rank of the chosen card is Ace"}, \\ D &= \text{"the suit of the chosen card is diamonds"}. \end{aligned}$$

  Are the events $C$ and $D$ independent? Again, justify your answer.

**Question 7:** Let $n \geq 2$ be an integer. Assume we have $n$ balls and 10 boxes. We throw the balls independently and uniformly at random in the boxes. Thus, for each $k$ and $i$ with $1 \leq k \leq n$ and $1 \leq i \leq 10$,

$$\Pr(\text{ the } k\text{-th ball falls in the } i\text{-th box }) = 1/10.$$

Define the event

$$A_n = \text{ "there is a box that contains at least two balls"}$$

and let $p_n = \Pr(A_n)$.

- Determine the smallest value of $n$ for which $p_n \geq 1/2$.
- Determine the smallest value of $n$ for which $p_n \geq 2/3$.

**Question 8:** Nick[3] is taking the course SPID 2804 (The Effect of Spiderman on the Banana Industry). The final exam for this course consists of one true/false question. To answer this question, Nick uses the following approach:

1. If Nick knows that the answer to the question is "true", he answers "true".
2. If Nick knows that the answer is "false", he answers "false".
3. If Nick does not know the answer, he flips a fair coin.
   (a) If the coin comes up heads, he answers "true".
   (b) If the coin comes up tails, he answers "false".

You are given that Nick knows the answer to the question with probability 0.8. Define the event

$$A = \text{“Nick gives the correct answer to the question”}.$$

- Determine $\Pr(A)$.

  *Hint:* Use the event $B$ = "Nick knows the answer". What are the conditional probabilities $\Pr(A \mid B)$ and $\Pr\left(A \mid \overline{B}\right)$?

**Question 9:** You are asked to design a random bit generator. You find a coin in your pocket, but, unfortunately, you are not sure if it is a fair coin. After some thought, you come up with the following algorithm GenerateBit($n$), which takes as input an integer $n \geq 1$:

**Algorithm** GenerateBit($n$)**:**

```
// all coin flips made are mutually independent
flip the coin n times;
k = the number of heads in the sequence of n coin flips;
if k is odd
then return 0
else return 1
endif
```

In this exercise, you will show that, when $n \to \infty$, algorithm GenerateBit($n$) returns a uniformly random bit.

Let $p$ be the real number with $0 < p < 1$, such that, if the coin is flipped once, it comes up heads with probability $p$ and tails with probability $1 - p$. (Note that algorithm GenerateBit does not need to know the value of $p$.) For any integer $n \geq 1$, define the two events

$$A_n = \text{“algorithm GenerateBit}(n)\text{ returns 0”}$$

[3] your friendly TA

and

$$B_n = \text{"the } n\text{-th coin flip made by algorithm } \textsc{GenerateBit}(n) \text{ results in heads"},$$

and define

$$P_n = \Pr(A_n)$$

and

$$Q_n = P_n - 1/2.$$

- Determine $P_1$ and $Q_1$.
- For any integer $n \geq 2$, prove that

  $$P_n = p + (1 - 2p) \cdot P_{n-1}.$$

  *Hint:* Express the event $A_n$ in terms of the events $A_{n-1}$ and $B_n$.
- For any integer $n \geq 2$, prove that

  $$Q_n = (1 - 2p) \cdot Q_{n-1}.$$
- For any integer $n \geq 1$, prove that

  $$Q_n = (1 - 2p)^{n-1} \cdot (p - 1/2).$$
- Prove that

  $$\lim_{n \to \infty} Q_n = 0$$

  and

  $$\lim_{n \to \infty} P_n = 1/2.$$

**Question 10:** Let $p$ be a real number with $0 < p < 1$. You are given two coins $C_1$ and $C_2$. The coin $C_1$ is fair, i.e., if you flip this coin, it comes up heads with probability 1/2 and tails with probability 1/2. If you flip the coin $C_2$, it comes up heads with probability $p$ and tails with probability $1 - p$. You pick one of these two coins uniformly at random, and flip it twice. These two coin flips are independent of each other. Define the events

$$\begin{aligned} A &= \text{"the first coin flip results in heads"}, \\ B &= \text{"the second coin flip results in heads"}. \end{aligned}$$

- Determine $\Pr(A)$.

  *Hint:* Express $\Pr(A)$ in terms of conditional probabilities, depending on which coin is chosen.
- Assume that $p = 1/4$. Are the events $A$ and $B$ independent? As always, justify your answer.
- Determine all values of $p$ for which the events $A$ and $B$ are independent. Again, justify your answer.