COMP 2804 — Assignment 4

Due: Thursday April 5, before 11:55pm, through cuLearn.

Assignment Policy:

- Your assignment must be submitted as one single PDF file through cuLearn.
- Late assignments will not be accepted.
- You are encouraged to collaborate on assignments, but at the level of discussion only. When writing your solutions, you must do so in your own words.
- Past experience has shown conclusively that those who do not put adequate effort into the assignments do not learn the material and have a probability near 1 of doing poorly on the exams.
- When writing your solutions, you must follow the guidelines below.
  - You must justify your answers.
  - The answers should be concise, clear and neat.
  - When presenting proofs, every step should be justified.

Question 1: Write your name and student number.

Question 2: You are given a fair red die and a fair blue die. You roll each die once, independently of each other. Let (i, j) be the outcome, where i is the result of the red die and j is the result of the blue die. Define the random variables

$$X = i + j$$

and

$$Y = i - j.$$

- Are X and Y independent random variables? As always, justify your answer.

Question 3: You are given two independent random variables X and Y, where

$$\Pr(X = 1) = \Pr(X = -1) = \Pr(Y = 1) = \Pr(Y = -1) = 1/2.$$

Define the random variable $Z = X \cdot Y$.

- Are X and Z independent random variables? As always, justify your answer.

Question 4: Alexa[1] and Shelly[2] take turns flipping, independently, a coin, where Alexa starts. The game ends as soon as heads comes up. The lady who flips heads first is the winner of the game.

Alexa proposes that they both use a fair coin. Of course, Shelly does not agree, because she knows that this gives Alexa a probability of $2/3$ of winning the game.

The ladies agree on the following: Let p and q be real numbers with $0 < p < 1$ and $0 \leq q \leq 1$. Alexa uses a coin that comes up heads with probability p, and Shelly uses a coin that comes up heads with probability q.

- Assume that $p = 1/2$. Determine the value of q for which Alexa and Shelly have the same probability of winning the game. A few sentences are sufficient to explain your answer.
- From now on, assume that $0 < q < 1$.
  - Determine the probability that Alexa wins the game.
  - Assume that $p > 1/2$. Prove that for any q with $0 < q < 1$, the probability that Alexa wins the game is strictly larger than $1/2$.
  - Assume that $p < 1/2$. Determine the value of q for which Alexa and Shelly have the same probability of winning the game.

Question 5: You roll a fair die five times, where all rolls are independent of each other. Define the random variable

$$X = \text{ the largest value in these five rolls.}$$

- Prove that the expected value $\mathbb{E}(X)$ of the random variable X is equal to

$$\mathbb{E}(X) = \frac{14077}{2592}.$$

Hint: What are the possible value for X? What is $\Pr(X = k)$? Use Wolfram Alpha to compute your final answer.

[1]your friendly TA
[2]another friendly TA

Question 6: Consider the following algorithm, which takes as input a large integer n and returns a random subset A of the set $\{1, 2, \ldots, n\}$:

```
Algorithm RandomSubset(n):

    // all coin flips made are mutually independent
    A = ∅;
    for i = 1 to n
    do flip a fair coin;
        if the result of the coin flip is heads
        then A = A ∪ {i}
        endif
    endfor;
    return A
```

Define

$$\max(A) = \begin{cases} \text{the largest element in } A & \text{if } A \neq \emptyset, \\ 0 & \text{if } A = \emptyset, \end{cases}$$

$$\min(A) = \begin{cases} \text{the smallest element in } A & \text{if } A \neq \emptyset, \\ 0 & \text{if } A = \emptyset, \end{cases}$$

and the random variable

$$X = \max(A) - \min(A).$$

- Prove that the expected value $\mathbb{E}(X)$ of the random variable X satisfies

  $$\mathbb{E}(X) = n - 3 + f(n),$$

  where $f(n)$ is some function that converges to 0 when $n \to \infty$.

  Hint: Introduce random variables $Y = \min(A)$ and $Z = \max(A)$ and compute their expected values. You may use

  $$\sum_{k=1}^{n} k \cdot x^k = \frac{x \left(n \cdot x^{n+1} - (n+1) \cdot x^n + 1\right)}{(x-1)^2}.$$

- Give an intuitive explanation, in a few sentences, why $\mathbb{E}(X)$ is approximately equal to $n - 3$.

Question 7: Let $m \geq 1$ and $n \geq 1$ be integers. You are given m cider bottles $C_1, C_2, \ldots, C_m$ and n beer bottles $B_1, B_2, \ldots, B_n$. Consider a uniformly random permutation of these $m+n$ bottles. The positions in this permutation are numbered $1, 2, \ldots, m+n$. Define the random variable

$$X = \text{ the position of the leftmost cider bottle.}$$

- Determine the possible values for X.
- For any value k that X can take, prove that

$$\Pr(X = k) = \frac{m}{k} \cdot \frac{\binom{n}{k-1}}{\binom{m+n}{k}}.$$

  Hint: Use the Product Rule to determine the number of permutations for which $X = k$. Rewrite your answer using basic properties of binomial coefficients.

- For each $i = 1, 2, \ldots, n$, define the indicator random variable

$$X_i = \begin{cases} 1 & \text{if } B_i \text{ is to the left of all cider bottles,} \\ 0 & \text{otherwise.} \end{cases}$$

  Prove that

$$\mathbb{E}(X_i) = \frac{1}{m+1}.$$

- Express X in terms of $X_1, X_2, \ldots, X_n$.
- Use the expression from the previous part to determine $\mathbb{E}(X)$.
- Prove that

$$\sum_{k=1}^{n+1} \frac{\binom{n}{k-1}}{\binom{m+n}{k}} = \frac{m+n+1}{m(m+1)}.$$

Question 8: You roll a fair die repeatedly, and independently, until you have seen all of the numbers $1, 2, 3, 4, 5, 6$ at least once. Define the random variable

$$X = \text{ the number of times you roll the die.}$$

For example, if you roll the sequence

$$5, 5, 3, 5, 1, 3, 4, 2, 5, 2, 1, 3, 6,$$

then $X = 13$.

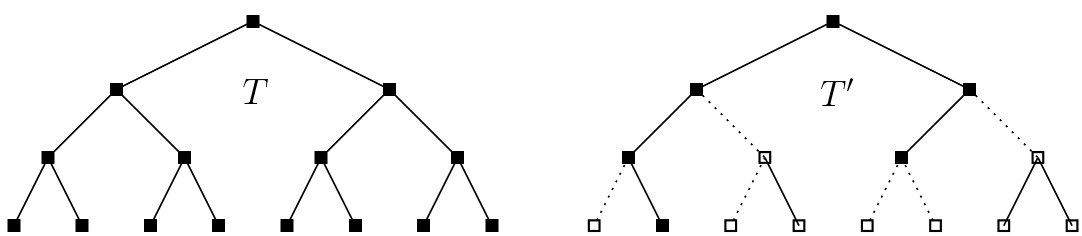
Determine the expected value $\mathbb{E}(X)$ of the random variable X.

Hint: Use the Linearity of Expectation. If you have seen exactly i different elements from the set $\{1, 2, 3, 4, 5, 6\}$, how many times do you expect to roll the die until you see a new element from this set?

Question 9: Let $k \geq 0$ be an integer and let T be a full binary tree, whose levels are numbered $0, 1, 2, \ldots, k$. (The root is at level 0, whereas the leaves are at level k.) Assume that each edge of T is removed with probability $1/2$, independently of other edges. Denote the resulting graph by T'.

Define the random variable X to be the number of nodes that are connected to the root by a path in T'; the root itself is included in X.

In the left figure below, the tree T is shown for the case when $k = 3$. The right figure shows the tree T': The dotted edges are those that have been removed from T, the black nodes are connected to the root by a path in T', whereas the white nodes are not connected to the root by a path in T'. For this case, $X = 6$.



- Let n be the number of nodes in the tree T. Express n in terms of k.
- Prove that the expected value $\mathbb{E}(X)$ of the random variable X is equal to

$$\mathbb{E}(X) = \log(n + 1).$$

Hint: For any ℓ with $0 \leq \ell \leq k$, how many nodes of T are at level ℓ? Use indicator random variables to determine the expected number of level-ℓ nodes of T that are connected to the root by a path in T'.