COMP 2804 — Solutions Assignment 1

Question 1:

- Write your name and student number.

Solution:

- Name: James Bond

- Student number: 007

Question 2: Let S be the set of all integers $x > 6543$ such that the decimal representation of x has distinct digits, none of which is equal to 7, 8, or 9. (The decimal representation does not have leading zeros.) Determine the size of the set S.

(You do not get marks if you write out all elements of S.)

Solution: We start by considering the number of digits that elements of S can have:

- Any integer with 1, 2, or 3 digits is less than 6543. Thus, elements of S cannot have 3, or fewer, digits.

- The largest 4-digit integer having distinct digits, none of which is equal to 7, 8, or 9, is 6543. Thus, elements of S cannot have 4 digits.

- We can only use the 7 digits $0, 1, \ldots, 6$. By the Pigeonhole Principle, any integer with 8, or more, digits must contain duplicate digits. Thus, elements of S cannot have 8 or more, digits.

- Conclusion: Each element of S has 5, 6, or 7 digits. We are going to count these separately.

- How many elements of S have 5 digits:

  - For the first digit, there are 6 choices: one of $1, 2, 3, 4, 5, 6$; note that we cannot choose 0 for the first digit.
  - For the second digit, there are 6 choices: any one of $0, 1, 2, 3, 4, 5, 6$, except the one we choose for the first digit.
  - For the third digit, there are 5 choices: any one of $0, 1, 2, 3, 4, 5, 6$, except the ones we choose for the first and second digits.
  - For the fourth digit, there are 4 choices.
  - For the fifth digit, there are 3 choices.

– Conclusion: The number of 5-digit elements of S is equal to

$$6 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 2160.$$

• By the same reasoning, the number of 6-digit elements of S is equal to

$$6 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 4320.$$

• By the same reasoning, the number of 7-digit elements of S is equal to

$$6 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 4320.$$

• Conclusion: The number of elements in S is equal to

$$2160 + 4320 + 4320 = 10,800.$$

Question 3: Let S be the set of all integers $x \in \{1, 2, \ldots, 100\}$ such that the decimal representation of x does not contain the digit 4. (The decimal representation does not have leading zeros.)

• Determine the size of the set S without using the Complement Rule.

• Use the Complement Rule to determine the size of the set S.

(You do not get marks if you write out all numbers from 1 to 100 and mark those that belong to the set S.)

Solution: We start by determining the size of S directly, i.e., without using the Complement Rule:

• How many elements in S have 1 digit: This digit can be any of the digits $1, 2, 3, 5, 6, 7, 8, 9$. Thus, the answer is 8.

• How many elements in S have 2 digits:

– The first digit can be any of the digits $1, 2, 3, 5, 6, 7, 8, 9$. Thus, there are 8 choices for the first digit. (In other words, for the first digit you can choose any of the 10 digits, except 0 and 4.)

– The second digit can be any of the digits $0, 1, 2, 3, 5, 6, 7, 8, 9$; thus any of the 10 digits, except 4. Thus, there are 9 choices for the first digit.

– Thus, the number of 2-digit elements of S is equal to $8 \cdot 9 = 72$.

• How many elements in S have 3 digits: There is only 1 such element, namely 100.

• Conclusion: The size of the set S is equal to $8 + 72 + 1 = 81$.

Now we are going to use the Complement Rule (and hope that we get the same answer!):

- We know that $|S| = 100 - |\overline{S}|$.

- Note that $x \in \overline{S}$ if and only if $1 \le x \le 100$ and the decimal representation of x contains at least one 4.

- How many elements in $\overline{S}$ have 1 digit: There is only 1 such element, namely 4.

- How many elements in $\overline{S}$ have 2 digits:

  - If the first digit is 4, then there are 10 choices for the second digit. Thus, $\overline{S}$ contains 10 many 2-digit elements that start with 4. Note that this includes the element 44.

  - If the first digit is not 4, whereas the second digit is 4: The first digit can be any of the digits $1, 2, 3, 5, 6, 7, 8, 9$. Thus, this gives 8 elements of $\overline{S}$.

  - Thus, the number of 2-digit elements in $\overline{S}$ is equal to $10 + 8 = 18$.

- How many elements in $\overline{S}$ have 3 digits: The answer is 0.

- Conclusion: The size of the set $\overline{S}$ is equal to $1 + 18 + 0 = 19$.

- Conclusion: $|S| = 100 - |\overline{S}| = 100 - 19 = 81$.

Question 4: The Ottawa Senators and the Toronto Maple Leafs play a best-of-7 series: These two hockey teams play games against each other, and the first team to win 4 games wins the series. Each game has a winner (thus, no game ends in a tie).

A sequence of games can be described by a string consisting of the characters S (indicating that the Senators win the game) and L (indicating that the Leafs win the game). Two possible ways for the Senators to win the series are (L, S, S, S, S) and (S, L, S, L, S, S).

Determine the number of ways in which the Senators can win the series.

(You do not get marks if you write out all possible ways.)

Solution: Here is a first solution. The number of games in any series is equal to 4, 5, 6, or 7. We are going to count all these types of series separately:

- How many series have 4 games: This is only possible if the Sens win the first 4 games. Thus, there is 1 such series.

- How many series have 5 games: A series has 5 games if and only if the Sens win 3 out of the first 4 games and they win the 5-th game. Thus, the number of such series is equal to $\binom{4}{3} = 4$.

- How many series have 6 games: A series has 6 games if and only if the Sens win 3 out of the first 5 games and they win the 6-th game. Thus, the number of such series is equal to $\binom{5}{3} = 10$.

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- How many series have 7 games: A series has 7 games if and only if the Sens win 3 out of the first 6 games and they win the 7-th game. Thus, the number of such series is equal to $\binom{6}{3} = 20$.

- Conclusion: The number of ways in which the Senators can win the series is equal to $1 + 4 + 10 + 20 = 35$.

Here is a second solution: We have seen above that a series can have 4, 5, 6, or 7 games. A series stops as soon as one team has won 4 games. *Imagine* that a series *always* has 7 games: If a team has won 4 games and the number of games played is less than 7, then the teams will play additional games and the winner of the series loses all of them.

In this way, each series has exactly 7 games. The Sens win the series if and only if they win exactly 4 games out of these 7 games. This gives as answer $\binom{7}{4}$, which is 35.

Let us make this more precise. Let A be the set of all series that are won by the Sens and that end as soon as the Sens have won 4 games. Let B be the set of all 7-game series in which the Sens win exactly 4 times. Consider the following function $f : A \to B$:

Let $(x_1, x_2, \ldots, x_k)$ be an element in A. This means that each x_i is one of the symbols S and L, the number of S's is equal to 4, the last symbol, i.e., x_k, is equal to S, and $k \leq 7$. Then

$$f(x_1, x_2, \ldots, x_k) = (x_1, x_2, \ldots, x_k, \underbrace{L, L, \ldots, L}_{7-k}).$$

For example,

$$f(S, S, S, S) = (S, S, S, S, L, L, L),$$
$$f(L, S, S, S, S) = (L, S, S, S, S, L, L),$$
$$f(S, L, S, L, S, S) = (S, L, S, L, S, S, L),$$

and

$$f(L, L, L, S, S, S, S) = (L, L, L, S, S, S, S).$$

This function is a bijection:

- The function is one-to-one: If you take two different elements in A, then they are mapped to two different elements in B.

- The function is onto: If you take an arbitrary element in B, then there is an element in A that maps to it.

Since $f : A \to B$ is a bijection, we have

$$|A| = |B|.$$

It is clear that $|B| = \binom{7}{4} = 35$. It follows that the size of A is 35 as well. (Note that the question asks for the size of A.)

Question 5: Let $m \geq 2$ and $n \geq 2$ be even integers. You are given m beer bottles $B_1, B_2, \ldots, B_m$ and n cider bottles $C_1, C_2, \ldots, C_n$. Assume you arrange these $m + n$ bottles on a horizontal line such that

- the leftmost $m/2$ bottles are all beer bottles, and

- the rightmost $n/2$ bottles are all cider bottles.

How many such arrangements are there? (The order of the bottles matters.)

Solution: We are going to use the Product Rule:

- Choose $m/2$ beer bottles. There are $\binom{m}{m/2}$ ways to do this.

- Place the chosen beer bottles at the first $m/2$ positions. There are $(m/2)!$ ways to do this.

- Choose $n/2$ cider bottles. There are $\binom{n}{n/2}$ ways to do this.

- Place the chosen cider bottles at the last $n/2$ positions. There are $(n/2)!$ ways to do this.

- We are left with $(m+n)/2$ bottles. Place these at the remaining positions. There are $((m+n)/2)!$ ways to do this.

By the Product Rule, the number of arrangements is equal to

$$\binom{m}{m/2} \cdot (m/2)! \cdot \binom{n}{n/2} \cdot (n/2)! \cdot ((m+n)/2)!.$$

Using

$$\binom{a}{b} = \frac{a!}{b!(a-b)!},$$

this is the same as

$$\frac{m!}{(m/2)!} \cdot \frac{n!}{(n/2)!} \cdot \left(\frac{m+n}{2}\right)!.$$

Question 6: Consider strings consisting of 40 characters, where each character is one of the letters a, b, and c. Such a string is called *cool* if it contains exactly 8 many a's or exactly 7 many b's. Determine the number of cool strings.

Solution: Let A be the set of all strings of length 40 that contain exactly 8 many a's, and let B be the set of all strings of length 40 that contain exactly 7 many b's. We have to determine the size of the set $A \cup B$.

- What is the size of the set A:

  - Choose 8 positions for the a's. There are $\binom{40}{8}$ ways to do this.
  - In each of the remaining 32 positions, write b or c. There are 2^{32} ways to do this.

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– By the Product Rule,
$$|A| = \binom{40}{8} \cdot 2^{32}.$$

- What is the size of the set B:

    – Choose 7 positions for the b's. There are $\binom{40}{7}$ ways to do this.
    – In each of the remaining 33 positions, write a or c. There are 2^{33} ways to do this.
    – By the Product Rule,
    $$|B| = \binom{40}{7} \cdot 2^{33}.$$

- What is the size of the set $A \cap B$:

    – Choose 8 positions for the a's. There are $\binom{40}{8}$ ways to do this.
    – Out of the remaining 32 positions, choose 7 for the b's. There are $\binom{32}{7}$ ways to do this.
    – In each of the remaining 25 positions, write c. There is 1 way to do this.
    – By the Product Rule,
    $$|A \cap B| = \binom{40}{8} \cdot \binom{32}{7}.$$

By using Inclusion-Exclusion, we get

$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B| \\ &= \binom{40}{8} \cdot 2^{32} + \binom{40}{7} \cdot 2^{33} - \binom{40}{8} \cdot \binom{32}{7}. \end{aligned}$$

Using Wolfram Alpha, this is equal to

$$490,191,217,850,773,920,$$

which is

490 quadrillion 191 trillion 217 billion 850 million 773 thousand 920.

Question 7: Consider a group of 100 students. In this group, 13 students like Donald Trump, 25 students like Justin Bieber, and 8 students like Donald Trump and like Justin Bieber. How many students in this group do not like Donald Trump and do not like Justin Bieber?

Solution: We will use the following sets:
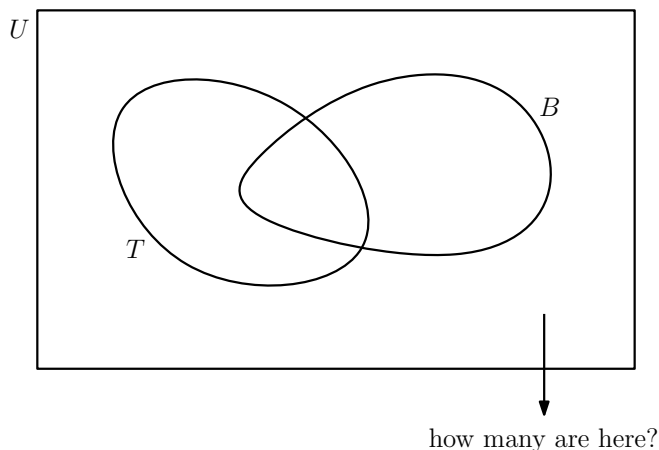
- U is the set of all 100 students in the group.

- T is the set of all students in U who like Donald Trump.

- B is the set of all students in U who like Justin Bieber.

The question asks to determine the size of the set

$$U \setminus (T \cup B),$$

which is equal to

$$|U| - |T \cup B|.$$

how many are here?

We are given the following:

- $|U| = 100$.

- $|T| = 13$.

- $|B| = 25$.

- $|T \cap B| = 8$.

Using Inclusion-Exclusion, we get

$$|T \cup B| = |T| + |B| - |T \cap B| = 13 + 25 - 8 = 30.$$

From this, the answer to the question is

$$|U| - |T \cup B| = 100 - 30 = 70.$$

Question 8: Use Newton's Binomial Theorem to prove that for every integer $n \geq 2$,

$$\sum_{k=0}^{n} \binom{n}{k} (n-1)^{n-k} = n^n. \tag{1}$$

In the rest of this exercise, you will give a combinatorial proof of this identity.

Consider the set $S = \{1, 2, \ldots, n\}$. We have seen in class that the number of functions $f : S \to S$ is equal to n^n.

- Consider a fixed integer k with $0 \le k \le n$ and a fixed subset A of S having size k. Determine the number of functions $f : S \to S$ having the property that $f(x) = x$ for all $x \in A$, and $f(x) \ne x$ for all $x \in S \setminus A$.

- Explain why the above part implies the identity in (1).

    Hint: Divide the functions f into groups based on the number of x for which $f(x) = x$.

Solution: Newton tells us that

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

If we take $x = n - 1$ and $y = 1$, then we get

$$((n - 1) + 1)^n = \sum_{k=0}^{n} \binom{n}{k} (n - 1)^{n-k} 1^k,$$

which is the same as

$$\sum_{k=0}^{n} \binom{n}{k} (n - 1)^{n-k} = n^n.$$

For the second part, we are going to use the Product rule: We obtain a function $f : S \to S$ having the property that $f(x) = x$ for all $x \in A$, and $f(x) \ne x$ for all $x \in S \setminus A$, in the following way:

- For each $x \in A$, we set $f(x) = x$. There is one way to do this.

- For each $x \in S \setminus A$, we set $f(x)$ to any value in $\{1, 2, \ldots, n\} \setminus \{x\}$. For each such x, there are $n - 1$ choices for $f(x)$. Since $|S \setminus A| = |S| - |A| = n - k$, there are $(n - 1)^{n-k}$ ways to do this for all $x \in S \setminus A$.

- Thus, the answer to this part of the question is $(n - 1)^{n-k}$.

For the third part, we know that the total number of functions $f : S \to S$ is equal to n^n. We are going to divide these functions into groups G_k, $k = 0, 1, 2, \ldots, n$:

Group G_k contains all functions $f : S \to S$ having the property that there are exactly k elements x for which $f(x) = x$.

If we denote the size of group G_k by N_k, then $\sum_{k=0}^{n} N_k$ is equal to the total number of functions. In other words, we have

$$\sum_{k=0}^{n} N_k = n^n.$$

We next determine N_k:

- Choose a k-element subset A of S. There are $\binom{n}{k}$ ways to do this.

- For the chosen subset A, choose a function f for which $f(x) = x$ for all $x \in A$, and $f(x) \neq x$ for all $x \in S \setminus A$. We have seen above that there are $(n-1)^{n-k}$ ways to do this.

- By the Product Rule, we get

$$N_k = \binom{n}{k}(n-1)^{n-k}.$$

Question 9: Let $n \geq 1$ be an integer. We consider binary $2 \times n$ matrices, i.e., matrices with 2 rows and n columns, in which each entry is 0 or 1. Any column in such a matrix is of one of four types, based on the bits that occur in this column. We will refer to these types as $\genfrac{}{}{0pt}{}{0}{0}$-columns, $\genfrac{}{}{0pt}{}{0}{1}$-columns, $\genfrac{}{}{0pt}{}{1}{0}$-columns, and $\genfrac{}{}{0pt}{}{1}{1}$-columns. For example, in the 2×7 matrix below, the first, second, and fifth columns are $\genfrac{}{}{0pt}{}{0}{1}$-columns, the third and seventh columns are $\genfrac{}{}{0pt}{}{1}{1}$-columns, the fourth column is a $\genfrac{}{}{0pt}{}{0}{0}$-column, and the sixth column is a $\genfrac{}{}{0pt}{}{1}{0}$-column.

0	0	1	0	0	1	1
1	1	1	0	1	0	1

For the rest of this exercise, let k be an integer with $0 \leq k \leq 2n$. A binary $2 \times n$ matrix is called *awesome*, if it contains exactly k many 0's.

- How many 1's are there in an awesome $2 \times n$ matrix?

- How many awesome $2 \times n$ matrices are there?

- Let i be an integer and consider an arbitrary awesome $2 \times n$ matrix M with exactly $n - i$ many $\genfrac{}{}{0pt}{}{1}{1}$-columns.

   - Prove that $\lceil k/2 \rceil \leq i \leq k$.
   - Determine the number of $\genfrac{}{}{0pt}{}{0}{1}$-columns plus the number of $\genfrac{}{}{0pt}{}{1}{0}$-columns in M.

- Let i be an integer. Prove that the number of awesome $2 \times n$ matrices with exactly $n - i$ many $\genfrac{}{}{0pt}{}{1}{1}$-columns is equal to

$$2^{2i-k}\binom{n}{n-i}\binom{i}{2i-k}.$$

- Use the above results to prove that

$$\sum_{i=\lceil k/2 \rceil}^{k} 2^{2i}\binom{n}{i}\binom{i}{k-i} = 2^{k}\binom{2n}{k}.$$

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Solution: For the first part: An awesome $2 \times n$ matrix has $2n$ entries, k of which are 0. Therefore, it contains $2n - k$ many 1's.

For the second part: We can think of an awesome $2 \times n$ matrix as a bitstring of length $2n$ with k many 0's. We have seen in class that there are $\binom{2n}{k}$ many of these.

For the third part, we consider an awesome $2 \times n$ matrix M with exactly $n - i$ many $\binom{1}{1}$-columns.

1. These $\binom{1}{1}$-columns contain $2(n - i)$ many 1's. We have seen above that the total number of 1's is equal to $2n - k$. Therefore,

$$2(n - i) \leq 2n - k,$$

which is equivalent to $2i \geq k$, which is equivalent to $i \geq k/2$. Since i is an integer, we conclude that $i \geq \lceil k/2 \rceil$.

2. There are i columns which are not $\binom{1}{1}$-columns. Each such column contains at least one 0. Thus, the total number of 0's in the matrix is at least i. We also know that the matrix contains exactly k many 0's. Therefore, $k \geq i$.

3. Let X denote the number of $\binom{0}{1}$-columns plus the number of $\binom{1}{0}$-columns in M. Then the matrix has $i - X$ many $\binom{0}{0}$-columns. The number of 0's in the matrix is equal to

$$X + 2(i - X) = 2i - X.$$

We also know that the number of 0's is equal to k. We conclude that

$$2i - X = k,$$

implying that
$$X = 2i - k.$$

For the fourth part, we are going to use the Product Rule:

1. Choose $n - i$ columns out of n columns, and make these $\binom{1}{1}$-columns. There are $\binom{n}{n-i}$ ways to do this.

2. Out of the remaining i columns, choose $2i - k$. There are $\binom{i}{2i-k}$ ways to do this.

3. For each of the $2i - k$ chosen columns chosen in the previous step, make it a $\binom{0}{1}$-column or a $\binom{1}{0}$-column. There are 2^{2i-k} ways to do this.

4. We are left with $k - i$ columns. We make these $\binom{0}{0}$-columns. There is one way to do this.

By the Product Rule, the number of awesome $2 \times n$ matrices with exactly $n - i$ many $\binom{1}{1}$-columns is equal to

$$\binom{n}{n - i} \binom{i}{2i - k} 2^{2i-k}.$$

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For the final part: We have seen above that the total number of awesome $2 \times n$ matrices is equal to $\binom{2n}{k}$. We are going to divide these matrices into groups. For any i, group G_i contains all awesome $2 \times n$ matrices with exactly $n - i$ many $\frac{1}{1}$-columns. We have seen above that we get one group G_i for each i with $\lceil k/2 \rceil \leq i \leq k$. Since each awesome matrix is in exactly one group, the total number of awesome matrices is equal to

$$\sum_{i=\lceil k/2 \rceil}^{k} |G_i|.$$

Thus,

$$\sum_{i=\lceil k/2 \rceil}^{k} |G_i| = \binom{2n}{k}.$$

Since

$$|G_i| = 2^{2i-k} \binom{n}{n-i} \binom{i}{2i-k},$$

we get

$$\sum_{i=\lceil k/2 \rceil}^{k} 2^{2i-k} \binom{n}{n-i} \binom{i}{2i-k} = \binom{2n}{k}.$$

Using $\binom{a}{b} = \binom{a}{a-b}$, we see that

$$\binom{n}{n-i} = \binom{n}{i}$$

and

$$\binom{i}{2i-k} = \binom{i}{k-i}.$$

This gives

$$\sum_{i=\lceil k/2 \rceil}^{k} 2^{2i-k} \binom{n}{i} \binom{i}{k-i} = \binom{2n}{k}.$$

If we multiply both sides by 2^k, we get

$$\sum_{i=\lceil k/2 \rceil}^{k} 2^{2i} \binom{n}{i} \binom{i}{k-i} = 2^k \binom{2n}{k}.$$

Question 10: Let $S_1, S_2, \ldots, S_{50}$ be a sequence consisting of 50 subsets of the set $\{1, 2, \ldots, 55\}$. Assume that each of these 50 subsets consists of at least seven elements.

Use the Pigeonhole Principle to prove that there exist two distinct indices i and j, such that the largest element in S_i is equal to the largest element in S_j.

Solution: Since each subset S_i contains at least seven elements, we have

$$\max(S_i) \in \{7, 8, 9, \ldots, 55\}.$$

In other words, there are 49 many possible values for $\max(S_i)$.

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- For each $k \in \{7, 8, 9, \ldots, 55\}$, there is one box with label k.

- For each $i = 1, 2, \ldots, 50$, we place the subset S_i in the box with label $\max(S_i)$.

- Thus, we place 50 subsets in 49 boxes. By the Pigeonhole Principle, there is a box that contains two subsets S_i and S_j. Since S_i and S_j are in the same box, we have $\max(S_i) = \max(S_j)$.