COMP 2804 — Solutions Assignment 2

Question 1:

- Write your name and student number.

Solution:

- Name: Daniel Alfredsson
- Student number: 11

Question 2: The function $f : \mathbb{N} \to \mathbb{Z}$ is defined by

$$\begin{array}{rcl} f(0) & = & 0, \\ f(n) & = & f(n-1) + (n^2 - n - 4) \cdot 2^{n-1} \quad \text{if } n \geq 1. \end{array}$$

- Determine $f(n)$ for $n = 0, 1, 2, 3, 4, 5$.
- Prove that for every integer $n \geq 0$,

$$f(n) = \left(n^2 - 3n\right) \cdot 2^n.$$

Solution: We are given that $f(0) = 0$. From the recurrence, with $n = 1$, we get

$$\begin{array}{rcl} f(1) & = & f(0) + \left(1^2 - 1 - 4\right) \cdot 2^{1-1} \\ & = & 0 - 4 \\ & = & -4. \end{array}$$

From the recurrence, with $n = 2$, we get

$$\begin{array}{rcl} f(2) & = & f(1) + \left(2^2 - 2 - 4\right) \cdot 2^{2-1} \\ & = & -4 - 2 \cdot 2 \\ & = & -8. \end{array}$$

From the recurrence, with $n = 3$, we get

$$\begin{array}{rcl} f(3) & = & f(2) + \left(3^2 - 3 - 4\right) \cdot 2^{3-1} \\ & = & -8 + 2 \cdot 4 \\ & = & 0. \end{array}$$

From the recurrence, with $n = 4$, we get

$$\begin{aligned} f(4) &= f(3) + \left(4^2 - 4 - 4\right) \cdot 2^{4-1} \\ &= 0 + 8 \cdot 8 \\ &= 64. \end{aligned}$$

From the recurrence, with $n = 5$, we get

$$\begin{aligned} f(5) &= f(4) + \left(5^2 - 5 - 4\right) \cdot 2^{5-1} \\ &= 64 + 16 \cdot 16 \\ &= 320. \end{aligned}$$

Next, we prove by induction that for every integer $n \geq 0$,

$$f(n) = \left(n^2 - 3n\right) \cdot 2^n.$$

The base case is when $n = 0$. In this case, the left-hand side is equal to $f(0)$, which is 0. The right-hand side is equal to $(0^2 - 3 \cdot 0) \cdot 2^0$, which is also 0. This proves the base case.

For the induction step, let $n \geq 1$ be an integer, and assume the claim is true for $n - 1$. Thus, we assume that

$$f(n-1) = \left((n-1)^2 - 3(n-1)\right) \cdot 2^{n-1}.$$

Using the recurrence, the induction hypothesis, and basic algebra, we get

$$\begin{aligned} f(n) &= f(n-1) + \left(n^2 - n - 4\right) \cdot 2^{n-1} \\ &= \left((n-1)^2 - 3(n-1)\right) \cdot 2^{n-1} + \left(n^2 - n - 4\right) \cdot 2^{n-1} \\ &= \left(n^2 - 5n + 4\right) \cdot 2^{n-1} + \left(n^2 - n - 4\right) \cdot 2^{n-1} \\ &= \left(2n^2 - 6n\right) \cdot 2^{n-1} \\ &= \left(n^2 - 3n\right) \cdot 2^n. \end{aligned}$$

This proves the induction step.

Question 3: You are asked to come up with an exam question about recurrences that is in the same style as Question 2. Thus, you write down some recurrence, which you then solve. Afterwards, you give the recurrence to the students and you give them the solution as well. The students must then prove that the given solution is indeed correct.

This is a painful process, because you must solve the recurrence yourself. Since you are lazy, you start with the following:

Exam Question:

The function $f : \mathbb{N} \to \mathbb{N}$ is defined by

$$\begin{aligned} f(0) &= XXX, \\ f(n) &= f(n-1) + YYY \quad \text{if } n \geq 1. \end{aligned}$$

Prove that for every integer $n \geq 0$,

$$f(n) = 7n^2 - 2n + 9.$$

- Complete the question, i.e., fill in XXX and YYY, so that you obtain a complete recurrence that has the given solution.

Solution: If the solution is $f(n) = 7n^2 - 2n + 9$, then the base case of the recurrence must be

$$f(0) = 7 \cdot 0^2 - 2 \cdot 0 + 9 = 9.$$

Thus, $XXX = 9$.

To obtain the recurrence, if the solution is

$$f(n) = 7n^2 - 2n + 9,$$

then

$$f(n-1) = 7(n-1)^2 - 2(n-1) + 9.$$

This gives

$$\begin{aligned} YYY &= f(n) - f(n-1) \\ &= \left(7n^2 - 2n + 9\right) - \left(7(n-1)^2 - 2(n-1) + 9\right) \\ &= \left(7n^2 - 2n + 9\right) - \left(7n^2 - 16n + 18\right) \\ &= 14n - 9. \end{aligned}$$

Question 4: The sequence of numbers a_n, for $n \geq 0$, is recursively defined as follows:

$$\begin{aligned} a_0 &= 5, \\ a_1 &= 3, \\ a_n &= 6 \cdot a_{n-1} - 9 \cdot a_{n-2} \quad \text{if } n \geq 2. \end{aligned}$$

- Determine a_n for $n = 0, 1, 2, 3, 4, 5$.
- Prove that for every integer $n \geq 0$,

$$a_n = (5 - 4n) \cdot 3^n.$$

Solution: We are given that $a_0 = 5$ and $a_1 = 3$. From the recurrence, with $n = 2$, we get

$$a_2 = 6 \cdot a_1 - 9 \cdot a_0 = 6 \cdot 3 - 9 \cdot 5 = -27.$$

From the recurrence, with $n = 3$, we get

$$a_3 = 6 \cdot a_2 - 9 \cdot a_1 = 6 \cdot (-27) - 9 \cdot 3 = -189.$$

From the recurrence, with $n = 4$, we get

$$a_4 = 6 \cdot a_3 - 9 \cdot a_2 = 6 \cdot (-189) - 9 \cdot (-27) = -891.$$

From the recurrence, with $n = 5$, we get

$$a_5 = 6 \cdot a_4 - 9 \cdot a_3 = 6 \cdot (-891) - 9 \cdot (-189) = -3645.$$

Next, we prove by induction that for every integer $n \geq 0$,

$$a_n = (5 - 4n) \cdot 3^n.$$

There are two base cases:

If $n = 0$: The left-hand side is equal to a_0, which is 5. The right-hand side is equal to $(5 - 4 \cdot 0) \cdot 3^0$, which is also 5.

If $n = 1$: The left-hand side is equal to a_1, which is 3. The right-hand side is equal to $(5 - 4 \cdot 1) \cdot 3^1$, which is also 3.

This proves the two base cases.

For the induction step, let $n \geq 2$ be an integer, and assume the claim is true for $n - 1$ and for $n - 2$. Thus, we assume that

$$a_{n-1} = (5 - 4(n-1)) \cdot 3^{n-1}$$

and

$$a_{n-2} = (5 - 4(n-2)) \cdot 3^{n-2}.$$

From the recurrence, these two assumptions, and basic algebra, we get

$$\begin{aligned} a_n &= 6 \cdot a_{n-1} - 9 \cdot a_{n-2} \\ &= 6\,(5 - 4(n-1)) \cdot 3^{n-1} - 9\,(5 - 4(n-2)) \cdot 3^{n-2} \\ &= 2\,(5 - 4(n-1)) \cdot 3^n - (5 - 4(n-2)) \cdot 3^n \\ &= 2\,(9 - 4n) \cdot 3^n - (13 - 4n) \cdot 3^n \\ &= (5 - 4n) \cdot 3^n. \end{aligned}$$

This proves the induction step.

Question 5: In this exercise, we consider strings of characters, where each character is an element of $\{a, b, c\}$. For any integer $n \geq 1$, let E_n be the number of such strings of length n that have an even number of c's, and let O_n be the number of such strings of length n that have an odd number of c's. (Recall that 0 is even.)

- Determine E_1, O_1, E_2, and O_2.
- Explain in plain English and at most two sentences why

$$E_n + O_n = 3^n.$$

- Prove that for every integer $n \geq 2$,

$$E_n = 2 \cdot E_{n-1} + O_{n-1}.$$

- Prove that for every integer $n \geq 1$,

$$E_n = \frac{1 + 3^n}{2}.$$

Solution: For $n = 1$, there are three strings of length 1: Both the strings a and b have an even number of c's, whereas the string c has an odd number of c's. This gives

$$E_1 = 2 \text{ and } O_1 = 1.$$

For $n = 2$, there are $3^2 = 9$ strings:

- How many of these have an odd number of c's: Such strings have one c and one non-c: There are 2 choices for the position of the c. Once this position has been chosen, we write a or b in the other position. This gives

$$O_2 = 2 \cdot 2 = 4.$$

- The other $9 - 4 = 5$ strings have an even number of c's. This gives

$$E_2 = 5.$$

Let $n \geq 1$. For each of the n positions, there are three possible characters. Therefore, the total number of strings is equal to 3^n. For each of these strings, the number of c's is either even or odd (and not both). Because of this,

$$E_n + O_n = 3^n.$$

Let $n \geq 2$. Consider all strings of length n with an even number of c's. There are E_n many of these. We divide these into three groups:

- The first group consists of all these strings that start with a. If we remove the first character from all strings in this group, then we obtain all strings of length $n - 1$ with an even number of c's. Thus, the number of strings in this group is equal to E_{n-1}.

- The second group consists of all these strings that start with b. If we remove the first character from all strings in this group, then we obtain all strings of length $n-1$ with an even number of c's. Thus, the number of strings in this group is equal to E_{n-1}.
- The third group consists of all these strings that start with c. If we remove the first character from all strings in this group, then we obtain all strings of length $n-1$ with an odd number of c's. Thus, the number of strings in this group is equal to O_{n-1}.

Since the groups are pairwise disjoint, we conclude that

$$E_n = E_{n-1} + E_{n-1} + O_{n-1} = 2 \cdot E_{n-1} + O_{n-1}.$$

For the last part of the question, we use the results that we have obtained above:

$$\begin{aligned} E_n &= 2 \cdot E_{n-1} + O_{n-1} \\ &= 2 \cdot E_{n-1} + \left(3^{n-1} - E_{n-1}\right) \\ &= E_{n-1} + 3^{n-1}. \end{aligned}$$

This gives a recurrence for the numbers E_n; recall that the base case is $E_1 = 2$. We use this recurrence to prove by induction that for every integer $n \geq 1$,

$$E_n = \frac{1+3^n}{2}.$$

The base case is when $n = 1$. The left-hand side is equal to E_1, which is 2. The right-hand side is equal to $\frac{1+3^1}{2}$, which is also 2. This proves the base case.

For the induction astep, let $n \geq 2$ be an integer, and assume that the claim is true for $n-1$. Thus, we assume that

$$E_{n-1} = \frac{1+3^{n-1}}{2}.$$

We now prove that the claim is also true for n:

$$\begin{aligned} E_n &= E_{n-1} + 3^{n-1} \\ &= \frac{1+3^{n-1}}{2} + 3^{n-1} \\ &= \frac{1+3^{n-1}}{2} + \frac{2 \cdot 3^{n-1}}{2} \\ &= \frac{1+3 \cdot 3^{n-1}}{2} \\ &= \frac{1+3^n}{2}. \end{aligned}$$

This proves the induction step.

Question 6: A *block* in a bitstring is a maximal consecutive substring of 1's. For example, the bitstring 1100011110100111 has four blocks: 11, 1111, 1, and 111.

For a given integer $n \geq 1$, consider all 2^n bitstrings of length n. Let B_n be the total number of blocks in all these bitstrings.

For example, the left part of the table below contains all 8 bitstrings of length 3. Each entry in the rightmost column shows the number of blocks in the corresponding bitstring. Thus,

$$B_3 = 0 + 1 + 1 + 1 + 1 + 2 + 1 + 1 = 8.$$

0	0	0	0
0	0	1	1
0	1	0	1
1	0	0	1
0	1	1	1
1	0	1	2
1	1	0	1
1	1	1	1

- Determine B_1 and B_2.

  Solution: We use the same notation as above. For $n = 1$, we get

0	0
1	1

  This shows that $B_1 = 0 + 1 = 1$.

  For $n = 2$, we get

0	0	0
0	1	1
1	0	1
1	1	1

  This shows that $B_2 = 0 + 1 + 1 + 1 = 3$.

- Let $n \geq 3$ be an integer.

  – Consider all bitstrings of length n that start with 0. What is the total number of blocks in these bitstrings?

    Solution: The number of bitstrings of length n that start with 0 is equal to 2^{n-1}. We want to know the total number of blocks in all these strings. If we remove the first bit from each of these strings, then the number of blocks does not change; moreover, this gives all bitstrings of length $n - 1$. Therefore, the answer is

    $$B_{n-1}. \tag{1}$$

– Determine the number of blocks in the bitstring

$$\underbrace{1\cdots1}_{n}.$$

Solution: This bitstring is one block. Therefore, the answer is

$$1. \tag{2}$$

– Determine the number of blocks in the bitstring

$$\underbrace{1\cdots1}_{n-1}0.$$

Solution: This bitstring contains one block. Therefore, the answer is

$$1. \tag{3}$$

– Let k be an integer with $2 \leq k \leq n-1$. Consider all bitstrings of length n that start with

$$\underbrace{1\cdots1}_{k-1}0.$$

Prove that the total number of blocks in these bitstrings is equal to

$$2^{n-k} + B_{n-k}.$$

Solution: How many strings are there of this type: The strings have length n, and the first k bits are fixed. Therefore, there are 2^{n-k} strings of this type. Each such string starts with a block of length $k-1$; this block is separated by a 0 from the other blocks in the string. This explains the term 2^{n-k}. If we remove the first k bits from all these strings, then we obtain all 2^{n-k} bitstrings of length $n-k$. The total number of blocks in these strings is equal to B_{n-k}; this explains the term B_{n-k}.

Therefore, the total number of blocks in all strings of this type is equal to

$$2^{n-k} + B_{n-k}. \tag{4}$$

– Prove that

$$B_n = 2 + B_{n-1} + \sum_{k=2}^{n-1} \left(2^{n-k} + B_{n-k}\right).$$

Solution: By definition, B_n is equal to the total number of blocks in all 2^n bitstrings of length n. In the previous parts, we have divided all these 2^n strings into groups; we determined the number of blocks within each group. Since the groups are pairwise disjoint, and together they contain all bitstrings of length n: If we add up all answers in (1)–(4), then the result is equal to B_n. Therefore, we have, for $n \geq 3$,

$$B_n = 2 + B_{n-1} + \sum_{k=2}^{n-1} \left(2^{n-k} + B_{n-k}\right). \tag{5}$$

– Use $1 + 2 + 2^2 + 2^3 + 2^{n-2} = 2^{n-1} - 1$, to prove that

$$B_n = 2^{n-1} + B_1 + B_2 + \cdots + B_{n-1}. \tag{6}$$

Solution: Using the hint, we get

$$\begin{aligned}\sum_{k=2}^{n-1} 2^{n-k} &= 2^{n-2} + 2^{n-3} + \cdots + 2^2 + 2 \\ &= \left(2^{n-2} + 2^{n-3} + \cdots + 2^2 + 2 + 1\right) - 1 \\ &= \left(2^{n-1} - 1\right) - 1 \\ &= 2^{n-1} - 2.\end{aligned}$$

Plugging this into (5) gives

$$\begin{aligned}B_n &= 2 + B_{n-1} + \sum_{k=2}^{n-1} \left(2^{n-k} + B_{n-k}\right) \\ &= 2 + B_{n-1} + \sum_{k=2}^{n-1} 2^{n-k} + \sum_{k=2}^{n-1} B_{n-k} \\ &= 2 + B_{n-1} + \left(2^{n-1} - 2\right) + \sum_{k=2}^{n-1} B_{n-k} \\ &= 2^{n-1} + B_{n-1} + \sum_{k=2}^{n-1} B_{n-k} \\ &= 2^{n-1} + B_1 + B_2 + \cdots + B_{n-1}.\end{aligned}$$

Remark: This gives a recurrence for the B_n's. Below, we will obtain a simpler recurrence.

- Prove that (6) also holds for $n = 2$.

  Solution: For $n = 2$, the left-hand side in (6) is B_2, which, as we have seen in the first part of this question, is equal to 3. The right-hand side in (6) is $2^{2-1} + B_1$, which, as we have seen in the first part of this question, is equal to $2 + 1 = 3$.

- Let $n \geq 3$. Prove that

  $$B_n = 2^{n-2} + 2 \cdot B_{n-1}.$$

  Hint: Write (6) on one line. Below this line, write (6) with n replaced by $n - 1$.

  Solution: We follow the hint: First, we write (6) for n. Then, we write (6) for $n - 1$; we can do this this, because $n - 1 \geq 2$.

  $$\begin{aligned}B_n &= 2^{n-1} + B_1 + B_2 + \cdots + B_{n-2} + B_{n-1}. \\ B_{n-1} &= 2^{n-2} + B_1 + B_2 + \cdots + B_{n-2}.\end{aligned}$$

If we subtract these equations, we get

$$\begin{aligned} B_n - B_{n-1} &= 2^{n-1} - 2^{n-2} + B_{n-1} \\ &= 2^{n-2} + B_{n-1}. \end{aligned}$$

This is equivalent to

$$B_n = 2^{n-2} + 2 \cdot B_{n-1}. \tag{7}$$

Remark: This gives a recurrence for the B_n's. I hope you agree that this recurrence is simpler than the one we had before.

- Prove that for every $n \geq 1$,

$$B_n = \frac{n+1}{4} \cdot 2^n.$$

Solution: We have seen in (7) that for $n \geq 3$,

$$B_n = 2^{n-2} + 2 \cdot B_{n-1}.$$

In fact, this is also true for $n = 2$: $B_2 = 3$ and $2^{2-2} + 2 \cdot B_{2-1} = 1 + 2 \cdot 1 = 3$.

We will use this recurrence to prove the claim.

The base case is when $n = 1$. In this case, the left-hand side is B_1, which is 1. The right-hand side is $\frac{1+1}{4} \cdot 2^1$, which is also 1. This proves the base case.

For the induction step, let $n \geq 2$, and assume the claim is true for $n - 1$. Thus, we assume that

$$B_{n-1} = \frac{n}{4} \cdot 2^{n-1}.$$

Using the recurrence, the induction hypothesis, and basic algebra, we get

$$\begin{aligned} B_n &= 2^{n-2} + 2 \cdot B_{n-1} \\ &= 2^{n-2} + 2 \cdot \frac{n}{4} \cdot 2^{n-1} \\ &= \frac{1}{4} \cdot 2^n + \frac{n}{4} \cdot 2^n \\ &= \frac{n+1}{4} \cdot 2^n. \end{aligned}$$

Remark: I am sure you all enjoyed this question. What a pain, eh? Near the end of the term, once we have seen indicator random variables, I will show you a *much* simpler solution!

Question 7: Let $n \geq 1$ be an integer and consider n beer bottles $B_1, B_2, \ldots, B_n$. In this exercise, we consider different ways to divide these bottles into subsets of size at most 2.

For example, if $n = 6$, then two different ways to do this are

$$\{B_1\}, \{B_2, B_6\}, \{B_3, B_4\}, \{B_5\}$$

and

$$\{B_1, B_4\}, \{B_2\}, \{B_3\}, \{B_5, B_6\}.$$

The order in which we write the subsets does not matter.

For each $n \geq 1$, let W_n be the number of different ways to divide n beer bottles into subsets of size at most 2.

- Determine W_1, W_2, W_3, and W_4.
- Prove that for every integer $n \geq 3$,

$$W_n = W_{n-1} + (n-1) \cdot W_{n-2}.$$

Solution: The following tables show all possible ways for $n = 1, 2, 3, 4$. Each row shows one way. The integers indicate the indices of the bottles.

$n = 1$
$\{1\}$
$W_1 = 1$

$n = 2$
$\{1\}$, $\{2\}$
$\{1, 2\}$
$W_2 = 2$

$n = 3$
$\{1\}$, $\{2\}$, $\{3\}$
$\{1\}$, $\{2, 3\}$
$\{1, 2\}$, $\{3\}$
$\{1, 3\}$, $\{2\}$
$W_3 = 4$

$n = 4$
$\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$
$\{1\}$, $\{2, 3\}$, $\{4\}$
$\{1\}$, $\{2, 4\}$, $\{3\}$
$\{1\}$, $\{3, 4\}$, $\{2\}$
$\{1, 2\}$, $\{3\}$, $\{4\}$
$\{1, 2\}$, $\{3, 4\}$
$\{1, 3\}$, $\{2\}$, $\{4\}$
$\{1, 3\}$, $\{2, 4\}$
$\{1, 4\}$, $\{2\}$, $\{3\}$
$\{1, 4\}$, $\{2, 3\}$
$W_4 = 10$

Next, we prove the recurrence. Let $n \geq 3$. By definition, there are W_n many ways to divide n bottles into subsets of size at most 2. We are going to divide these into 2 groups (the division can be seen in the tables for $n = 3$ and $n = 4$):

- The first group consists of all ways in which B_1 is in a subset of size 1. How many of these are there: If we remove B_1, then we obtain all ways to divide $n - 1$ bottles into groups of size at most 2. Thus, this group contains W_{n-1} ways.
- The second group consists of all ways in which B_1 is in a subset of size 2. How many of these are there:
  - There are $n - 1$ choices for the second bottle in B_1's subset.
  - For each choice, there are W_{n-2} ways to divide the remaining bottles into subsets of size at most 2.
  - By the Product Rule, the second group contains $(n - 1) \cdot W_{n-2}$ ways.

Since these two groups are disjoint, we conclude that

$$W_n = W_{n-1} + (n - 1) \cdot W_{n-2}.$$

Question 8: Consider the following recursive algorithm, which takes as input a sequence $(a_1, a_2, \ldots, a_n)$ of length n, where $n \geq 1$:

Algorithm $\text{MYSTERY}(a_1, a_2, \ldots, a_n)$**:**

 if $n = 1$
 then return the sequence (a_1)
 else $(b_1, b_2, \ldots, b_{n-1}) = \text{MYSTERY}(a_1, a_2, \ldots, a_{n-1})$;
  return the sequence $(a_n, b_1, b_2, \ldots, b_{n-1})$
 endif

- Express the output of algorithm $\text{MYSTERY}(a_1, a_2, \ldots, a_n)$ in terms of the input sequence $(a_1, a_2, \ldots, a_n)$. Prove that your answer correct.

Solution: After having stared at the algorithm long enough, you will guess that the output of a call to $\text{MYSTERY}(a_1, a_2, \ldots, a_n)$ is the reverse of the input sequence, i.e., $(a_n, a_{n-1}, \ldots, a_1)$. We will prove by induction on n that this is correct.

The base case is when $n = 1$. In this case, the output of $\text{MYSTERY}(a_1)$ is the sequence (a_1), which is indeed the reverse of the input.

For the induction step, let $n \geq 2$ be an integer, and assume that the claim is true for $n - 1$. Thus, we assume that, for any input sequence of length $n - 1$, algorithm MYSTERY returns the reverse sequence. We are going to show that the claim is true for n.

Let us see what algorithm $\text{MYSTERY}(a_1, a_2, \ldots, a_n)$ does:

- It runs MYSTERY($a_1, a_2, \ldots, a_{n-1}$). By the induction hypothesis, the output of this call is
$$(b_1, b_2, \ldots, b_{n-1}) = (a_{n-1}, a_{n-2}, \ldots, a_1).$$
- It returns the sequence $(a_n, b_1, b_2, \ldots, b_{n-1})$, which is equal to $(a_n, a_{n-1}, \ldots, a_1)$, which is the reverse of the input sequence.

This proves the induction step.

Question 9: Ever since he was a child, Nick[1] has been dreaming to be like Spiderman. As you all know, Spiderman can climb up the outside of a building; if he is at a particular floor, then, in one step, he can move up several floors. Nick is not that advanced yet. In one step, Nick can move up either one floor or two floors.

Let $n \geq 1$ be an integer and consider a building with n floors, numbered $1, 2, \ldots, n$. (The first floor has number 1; this is not the ground floor.) Nick is standing in front of this building, at the ground level. There are different ways in which Nick can climb to the n-th floor. For example, here are three different ways for the case when $n = 5$:

move up 2 floors, move up 1 floor, move up 2 floors.

move up 1 floor, move up 2 floors, move up 2 floors.

move up 1 floor, move up 2 floors, move up 1 floor, move up 1 floor .

Let S_n be the number of different ways, in which Nick can climb to the n-th floor.

- Determine, S_1, S_2, S_3, and S_4.
- Determine the value of S_n, i.e., express S_n in terms of numbers that we have seen in class. As always, justify your answer.

Solution:

1. For $n = 1$, there is only 1 way to move from the ground floor to the first floor. Thus, $S_1 = 1$.
2. For $n = 2$, here are the different ways to move from the ground floor to the second floor:
$$(1, 1), (2).$$
Thus, $S_2 = 2$.

[1] your friendly TA

3. For $n = 3$, here are the different ways to move from the ground floor to the third floor:

$$(1, 1, 1), (2, 1), (1, 2).$$

   Thus, $S_3 = 3$.

4. For $n = 4$, here are the different ways to move from the ground floor to the fourth floor:

$$(1, 1, 1, 1), (2, 1, 1), (1, 2, 1), (1, 1, 2), (2, 2).$$

   Thus, $S_4 = 5$.

Now we are going to derive a recurrence for the numbers S_n. Let $n \geq 2$. By definition, there are S_n many different ways for Nick to move from the ground floor to the n-th floor. We divide them into two groups, depending on the last step:

1. Group 1: In the last step, Nick moves from floor $n - 1$ to floor n.

   In each of these, Nick first has to move from the ground floor to floor $n - 1$. The number of ways to do this is equal to S_{n-1}.

2. Group 2: In the last step, Nick moves from floor $n - 2$ to floor n.

   In each of these, Nick first has to move from the ground floor to floor $n - 2$. The number of ways to do this is equal to S_{n-2}.

Since the two groups are disjoint, and together they contain all ways to move from the ground floor to floor n, it follows that

$$S_n = S_{n-1} + S_{n-2}.$$

This is of course, the Fibonacci recurrence.

In the following table, we compare the Fibonacci numbers f_n with the numbers S_n:

f_0	f_1	f_2	f_3	f_4	f_5	f_6
0	1	1	2	3	5	8
		S_1	S_2	S_3	S_4	S_5

From this table, we see that for each integer $n \geq 1$,

$$S_n = f_{n+1}.$$