COMP 2804 — Solutions Assignment 4

Question 1: Write your name and student number.

Solution:

- Name: Lieke Martens
- Student number: 22

Question 2: You are given a fair red die and a fair blue die. You roll each die once, independently of each other. Let (i, j) be the outcome, where i is the result of the red die and j is the result of the blue die. Define the random variables

$$X = i + j$$

and

$$Y = i - j.$$

- Are X and Y independent random variables? As always, justify your answer.

Solution: The random variables X and Y are independent if and only if for all a and for all b,

$$\Pr(X = a \wedge Y = b) = \Pr(X = a) \cdot \Pr(Y = b).$$

To show that X and Y are not independent, it suffices to find one value for a and one value for b, such that

$$\Pr(X = a \wedge Y = b) \neq \Pr(X = a) \cdot \Pr(Y = b).$$

We notice the following: If $X = 12$, then both i and j must be equal to 6 and, therefore, $Y = i - j = 0$. Based on this, we take $a = 12$ and $b = 1$. Since the event "$X = 12 \wedge Y = 1$" never happens, we have

$$\Pr(X = 12 \wedge Y = 1) = \Pr(\emptyset) = 0.$$

On the other hand,

$$\Pr(X = 12) = \Pr(6, 6) = 1/36$$

and

$$\Pr(Y = 1) = \Pr((6, 5) \vee (5, 4) \vee (4, 3) \vee (3, 2) \vee (2, 1)) = 5/36,$$

implying that

$$\Pr(X = 12) \cdot \Pr(Y = 1) \neq 0.$$

We conclude that X and Y are not independent.

Question 3: You are given two independent random variables X and Y, where

$$\Pr(X = 1) = \Pr(X = -1) = \Pr(Y = 1) = \Pr(Y = -1) = 1/2.$$

Define the random variable $Z = X \cdot Y$.

- Are X and Z independent random variables? As always, justify your answer.

Solution: We start by determining $\Pr(Z = 1)$: Since

$$Z = 1 \text{ if and only if } X = Y = 1 \text{ or } X = Y = -1,$$

we have

$$\begin{aligned} \Pr(Z = 1) &= \Pr((X = 1 \wedge Y = 1) \vee (X = -1 \wedge Y = -1)) \\ &= \Pr(X = 1 \wedge Y = 1) + \Pr(X = -1 \wedge Y = -1). \end{aligned}$$

Since X and Y are independent, we have

$$\Pr(X = 1 \wedge Y = 1) = \Pr(X = 1) \cdot \Pr(Y = 1) = 1/2 \cdot 1/2 = 1/4$$

and

$$\Pr(X = -1 \wedge Y = -1) = \Pr(X = -1) \cdot \Pr(Y = -1) = 1/2 \cdot 1/2 = 1/4.$$

We conclude that

$$\Pr(Z = 1) = 1/4 + 1/4 = 1/2.$$

Since $Z \in \{-1, 1\}$, we get

$$\Pr(Z = -1) = 1 - \Pr(Z = 1) = 1 - 1/2 = 1/2.$$

We will show that X and Z are independent. For this, we have to verify four equations:

1. Since

   $$X = 1 \wedge Z = 1 \text{ if and only if } X = 1 \wedge Y = 1,$$

   we have

   $$\Pr(X = 1 \wedge Z = 1) = \Pr(X = 1 \wedge Y = 1).$$

   Using the fact that X and Y are independent, we get

   $$\Pr(X = 1 \wedge Z = 1) = \Pr(X = 1) \cdot \Pr(Y = 1) = 1/2 \cdot 1/2 = 1/4.$$

   We also know that

   $$\Pr(X = 1) \cdot \Pr(Z = 1) = 1/2 \cdot 1/2 = 1/4.$$

   We conclude that

   $$\Pr(X = 1 \wedge Z = 1) = \Pr(X = 1) \cdot \Pr(Z = 1).$$

   This verifies the first equation.

2. Since
$$X = 1 \wedge Z = -1 \text{ if and only if } X = 1 \wedge Y = -1,$$
the same reasoning shows that
$$\Pr(X = 1 \wedge Z = -1) = \Pr(X = 1) \cdot \Pr(Z = -1).$$

3. Since
$$X = -1 \wedge Z = 1 \text{ if and only if } X = -1 \wedge Y = -1,$$
the same reasoning shows that
$$\Pr(X = -1 \wedge Z = 1) = \Pr(X = -1) \cdot \Pr(Z = 1).$$

4. Since
$$X = -1 \wedge Z = -1 \text{ if and only if } X = -1 \wedge Y = 1,$$
the same reasoning shows that
$$\Pr(X = -1 \wedge Z = -1) = \Pr(X = -1) \cdot \Pr(Z = -1).$$

Question 4: Alexa[1] and Shelly[2] take turns flipping, independently, a coin, where Alexa starts. The game ends as soon as heads comes up. The lady who flips heads first is the winner of the game.

Alexa proposes that they both use a fair coin. Of course, Shelly does not agree, because she knows that this gives Alexa a probability of 2/3 of winning the game.

The ladies agree on the following: Let p and q be real numbers with $0 < p < 1$ and $0 \leq q \leq 1$. Alexa uses a coin that comes up heads with probability p, and Shelly uses a coin that comes up heads with probability q.

- Assume that $p = 1/2$. Determine the value of q for which Alexa and Shelly have the same probability of winning the game. A few sentences are sufficient to explain your answer.
- From now on, assume that $0 < q < 1$.
  - Determine the probability that Alexa wins the game.
  - Assume that $p > 1/2$. Prove that for any q with $0 < q < 1$, the probability that Alexa wins the game is strictly larger than 1/2.
  - Assume that $p < 1/2$. Determine the value of q for which Alexa and Shelly have the same probability of winning the game.

[1] your friendly TA
[2] another friendly TA

Solution: We start with the case when $p = 1/2$. Alexa wins the game if her first coin flip results in heads; this happens with probability $1/2$. The only way for Alexa to have a winning probability of exactly $1/2$ is to have $q = 1$, i.e., Shelly's coin always comes up heads. In this case, Alexa wins the game if and only her first coin flip results in heads.

Next we assume that $0 < q < 1$. Alexa wins the game if and only if the first heads is on an odd-numbered coin flip. This implies that

$$\begin{aligned} \Pr(\text{ Alexa wins }) &= \sum_{n=0}^{\infty} \Pr\left((TT)^n H\right) \\ &= \sum_{n=0}^{\infty} \left((1-p)(1-q)\right)^n p \\ &= p \sum_{n=0}^{\infty} \left((1-p)(1-q)\right)^n . \end{aligned}$$

Recall that, for $|x| < 1$,

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

Therefore, we get (note that $0 < (1-p)(1-q) < 1$)

$$\Pr(\text{ Alexa wins }) = p \cdot \frac{1}{1-(1-p)(1-q))} = \frac{p}{p+q-pq}.$$

Now assume that $p > 1/2$. We have to show that

$$\Pr(\text{ Alexa wins }) > \frac{1}{2}.$$

This is equivalent to

$$\frac{p}{p+q-pq} > \frac{1}{2},$$

which is equivalent to

$$2p > p+q-pq,$$

which is equivalent to

$$p(1+q) > q,$$

which is equivalent to

$$p > \frac{q}{1+q}. \tag{1}$$

Now we have to convince ourselves that (1) holds. First note that

$$\frac{q}{1+q} < \frac{1}{2},$$

because this is equivalent to

$$2q < 1 + q,$$

which is equivalent to

$$q < 1,$$

which is true.

We conclude that

$$p > \frac{1}{2} > \frac{q}{1+q},$$

i.e., (1) holds.

We now assume that $p < 1/2$. We have seen before that

$$\Pr(\text{ Alexa wins }) = \frac{p}{p+q-pq}.$$

This is equal to $1/2$ if and only if

$$\frac{p}{p+q-pq} = \frac{1}{2},$$

which is equivalent to

$$2p = p + q - pq,$$

which is equivalent to

$$p = q(1-p),$$

which is equivalent to

$$q = \frac{p}{1-p}.$$

In order for this to be a valid solution, we still have to verify that $0 < q < 1$: It is obvious that $q > 0$.

We have $q < 1$ if and only if

$$\frac{p}{1-p} < 1,$$

which is equivalent to

$$p < 1 - p,$$

which is equivalent to

$$p < 1/2,$$

which is true.

Question 5: You roll a fair die five times, where all rolls are independent of each other. Define the random variable

$$X = \text{ the largest value in these five rolls.}$$

- Prove that the expected value $\mathbb{E}(X)$ of the random variable X is equal to

$$\mathbb{E}(X) = \frac{14077}{2592}.$$

Hint: What are the possible value for X? What is $\Pr(X = k)$? Use Wolfram Alpha to compute your final answer.

Solution: Since the possible values for X are $1, 2, 3, 4, 5, 6$, we have

$$\mathbb{E}(X) = \sum_{k=1}^{6} k \cdot \Pr(X = k).$$

The sample space is the set

$$S = \{(r_1, \ldots, r_5) : \text{ each } r_i \in \{1, 2, \ldots, 6\}\},$$

which has size 6^5. Since the rolls are independent, we have a uniform probability.

Let $1 \leq k \leq 6$. We have $X = k$ if and only if all rolls are from the set $\{1, 2, \ldots, k\}$ and at least one roll is equal to k. The number of ways for this to happen is equal to $k^5 - (k-1)^5$. (Observe that this is also true for $k = 1$.) It follows that

$$\Pr(X = k) = \frac{k^5 - (k-1)^5}{6^5}.$$

We conclude that

$$\mathbb{E}(X) = \sum_{k=1}^{6} k \cdot \left(\left(\frac{k}{6} \right)^5 - \left(\frac{k-1}{6} \right)^5 \right).$$

Wolfram Alpha will tell you that this summation is equal to $14077/2592$.

Question 6: Consider the following algorithm, which takes as input a large integer n and returns a random subset A of the set $\{1, 2, \dots, n\}$:

```
Algorithm RandomSubset(n):

    // all coin flips made are mutually independent
    A = ∅;
    for i = 1 to n
    do flip a fair coin;
        if the result of the coin flip is heads
        then A = A ∪ {i}
        endif
    endfor;
    return A
```

Define

$$\max(A) = \begin{cases} \text{the largest element in } A & \text{if } A \neq \emptyset, \\ 0 & \text{if } A = \emptyset, \end{cases}$$

$$\min(A) = \begin{cases} \text{the smallest element in } A & \text{if } A \neq \emptyset, \\ 0 & \text{if } A = \emptyset, \end{cases}$$

and the random variable

$$X = \max(A) - \min(A).$$

- Prove that the expected value $\mathbb{E}(X)$ of the random variable X satisfies

$$\mathbb{E}(X) = n - 3 + f(n),$$

  where $f(n)$ is some function that converges to 0 when $n \to \infty$.

  Hint: Introduce random variables $Y = \min(A)$ and $Z = \max(A)$ and compute their expected values. You may use

$$\sum_{k=1}^{n} k \cdot x^k = \frac{x\left(n \cdot x^{n+1} - (n+1) \cdot x^n + 1\right)}{(x-1)^2}.$$

- Give an intuitive explanation, in a few sentences, why $\mathbb{E}(X)$ is approximately equal to $n - 3$.

Solution: We start with the intuitive explanation. In class, we have seen the following: If we flip a fair and independent coin repeatedly until it comes up heads for the first time, then the expected number of flips is equal to 2.

The smallest element of the set A is obtained in the following way: For $i = 1, 2, \dots$, flip a fair and independent coin. The first time the coin comes up heads, the value of $\min(A)$ is set to the index i of this coin flip. If the coin never comes up heads during n coin flips, the value of $\min(A)$ is set to 0. If we assume that n is very large, then it is very likely that we flip heads during n coin flips. Because of this, the expected value of $\min(A)$ should be very close to 2.

For the largest element of the set A, we do the same thing, except that we flip the coin for $n, n-1, n-2, \ldots$. On average, the second flip will result in heads. Because of this, the expected value of $\max(A)$ should be very close to $n-1$.

Conclusion: The expected value of the random variable X should be very close to $(n-1)-2 = n-3$.

Now we give the details. We start with the expected value of $Y = \min(A)$. The possible values for Y are $0, 1, 2, \ldots, n$. Thus,

$$\mathbb{E}(Y) = \sum_{k=0}^{n} k \cdot \Pr(Y = k) = \sum_{k=1}^{n} k \cdot \Pr(Y = k).$$

Let $1 \leq k \leq n$. Then $Y = k$ if and only if the coin flips for $i = 1, 2, \ldots, k-1$ all result in tails and the coin flip for $i = k$ results in heads. It follows that $\Pr(Y = k) = (1/2)^k$ and, using the hint with $x = 1/2$, we get

$$\begin{aligned}
\mathbb{E}(Y) &= \sum_{k=1}^{n} k \cdot (1/2)^k \\
&= \frac{\frac{1}{2}\left(n \cdot (1/2)^{n+1} - (n+1) \cdot (1/2)^n + 1\right)}{(1/2-1)^2} \\
&= 2\left(n \cdot (1/2)^{n+1} - (n+1) \cdot (1/2)^n + 1\right) \\
&= 2 + n \cdot (1/2)^n - (n+1) \cdot (1/2)^{n-1}.
\end{aligned}$$

Next we determine the expected value of $Z = \max(A)$. The possible values for Z are $0, 1, 2, \ldots, n$. Thus,

$$\mathbb{E}(Z) = \sum_{k=0}^{n} k \cdot \Pr(Z = k) = \sum_{k=1}^{n} k \cdot \Pr(Z = k).$$

Let $1 \leq k \leq n$. Then $Z = k$ if and only if the coin flips for $i = n, n-1, \ldots, k+1$ all result in tails and the coin flip for $i = k$ results in heads. It follows that $\Pr(Z = k) = (1/2)^{n+1-k}$ and, using the hint with $x = 2$, we get

$$\begin{aligned}
\mathbb{E}(Z) &= \sum_{k=1}^{n} k \cdot (1/2)^{n+1-k} \\
&= (1/2)^{n+1} \sum_{k=1}^{n} k \cdot 2^k \\
&= (1/2)^{n+1} \cdot \frac{2\left(n \cdot 2^{n+1} - (n+1) \cdot 2^n + 1\right)}{(2-1)^2} \\
&= 2n - (n+1) + (1/2)^n \\
&= n - 1 + (1/2)^n.
\end{aligned}$$

Since $X = Z - Y$, we conclude that

$$\begin{aligned}
\mathbb{E}(X) &= \mathbb{E}(Z - Y) \\
&= \mathbb{E}(Z) - \mathbb{E}(Y) \\
&= (n - 1 + (1/2)^n) - \big(2 + n \cdot (1/2)^n - (n+1) \cdot (1/2)^{n-1}\big) \\
&= n - 3 + \big((1/2)^n - n \cdot (1/2)^n + (n+1) \cdot (1/2)^{n-1}\big) .
\end{aligned}$$

The entire term within the parentheses converges to 0 when $n \to \infty$.

Question 7: Let $m \geq 1$ and $n \geq 1$ be integers. You are given m cider bottles $C_1, C_2, \ldots, C_m$ and n beer bottles $B_1, B_2, \ldots, B_n$. Consider a uniformly random permutation of these $m+n$ bottles. The positions in this permutation are numbered $1, 2, \ldots, m+n$. Define the random variable

$$X = \text{ the position of the leftmost cider bottle.}$$

- Determine the possible values for X.
- For any value k that X can take, prove that

  $$\Pr(X = k) = \frac{m}{k} \cdot \frac{\binom{n}{k-1}}{\binom{m+n}{k}}.$$

  Hint: Use the Product Rule to determine the number of permutations for which $X = k$. Rewrite your answer using basic properties of binomial coefficients.

- For each $i = 1, 2, \ldots, n$, define the indicator random variable

  $$X_i = \begin{cases} 1 & \text{if } B_i \text{ is to the left of all cider bottles,} \\ 0 & \text{otherwise.} \end{cases}$$

  Prove that

  $$\mathbb{E}\left(X_i\right) = \frac{1}{m+1}.$$

- Express X in terms of $X_1, X_2, \ldots, X_n$.
- Use the expression from the previous part to determine $\mathbb{E}(X)$.
- Prove that

  $$\sum_{k=1}^{n+1} \frac{\binom{n}{k-1}}{\binom{m+n}{k}} = \frac{m+n+1}{m(m+1)}.$$

Solution: The possible values for X are $1, 2, \ldots, n+1$.

Let $1 \leq k \leq n+1$. We have $X = k$ if and only if the positions $1, 2, \ldots, k-1$ have beer bottles and the position k has a cider bottle. We are going to use the Product Rule to determine the number of permutations for which $X = k$:

1. Choose $k-1$ beer bottles. There are $\binom{n}{k-1}$ ways to do this.
2. Place the chosen beer bottles in an arbitrary order at the positions $1, 2, \ldots, k-1$. There are $(k-1)!$ ways to do this.
3. Choose a cider bottle and place it at position k. There are m ways to do this.
4. Place the remaining $m+n-k$ bottles in an arbitrary order at the positions $k+1, k+2, \ldots, m+n$. There are $(m+n-k)!$ ways to do this.

By the Product Rule, the number of permutations for which $X = k$ is equal to

$$\binom{n}{k-1} \cdot (k-1)! \cdot m \cdot (m+n-k)!.$$

Since we have a uniformly random permutation, it follows that

$$\begin{aligned}
\Pr(X = k) &= \frac{\binom{n}{k-1} \cdot (k-1)! \cdot m \cdot (m+n-k)!}{(m+n)!} \\
&= \frac{m}{k} \cdot \binom{n}{k-1} \cdot \frac{k! \cdot (m+n-k)!}{(m+n)!} \\
&= \frac{m}{k} \cdot \frac{\binom{n}{k-1}}{\binom{m+n}{k}}.
\end{aligned}$$

Note that this implies that

$$\begin{aligned}
\mathbb{E}(X) &= \sum_{k=1}^{n+1} k \cdot \Pr(X = k) \\
&= \sum_{k=1}^{n+1} m \cdot \frac{\binom{n}{k-1}}{\binom{m+n}{k}} \\
&= m \cdot \sum_{k=1}^{n+1} \frac{\binom{n}{k-1}}{\binom{m+n}{k}}. \qquad (2)
\end{aligned}$$

Since this expression looks too complicated, we are going to use indicator random variables to obtain a simpler expression for $\mathbb{E}(X)$.

Let $1 \leq i \leq n$. We know that

$$\mathbb{E}(X_i) = \Pr(X_i = 1) = \Pr(B_i \text{ to the left of } C_1, C_2, \ldots, C_m).$$

To determine this probability, we only care about the bottles $B_i, C_1, C_2, \ldots, C_m$. Since these bottles are in random order, the beer bottle B_i is at a uniformly random position in the permutation of these $m+1$ bottles. In particular, it is at the first position with probability $1/(m+1)$. Therefore,

$$\mathbb{E}(X_i) = \frac{1}{m+1}.$$

Alternatively, there are $(m+1)!$ permutations of the bottles $B_i, C_1, C_2, \ldots, C_m$. In exactly $m!$ of them, B_i is to the left of all cider bottles. Therefore,

$$\mathbb{E}(X_i) = \frac{m!}{(m+1)!} = \frac{1}{m+1}.$$

In case you prefer a more formal proof, we determine the number of permutations of all $m+n$ bottles for which $X_i = 1$:

1. Choose $n-1$ positions, out of $m+n$. There are $\binom{m+n}{n-1}$ ways to do this.
2. Place $B_1, \ldots, B_{i-1}, B_{i+1}, \ldots, B_n$ in the chosen positions. There are $(n-1)!$ ways to do this.
3. Place B_i in the leftmost empty position. There is 1 way to do this.
4. Place $C_1, \ldots, C_m$ in the remaining m positions. There are $m!$ ways to do this.

By the Product Rule, the number of permutations for which $X_i = 1$ is equal to

$$\begin{aligned} \binom{m+n}{n-1} \cdot (n-1)! \cdot m! &= \frac{(m+n)!}{(n-1)! \cdot (m+1)!} \cdot (n-1)! \cdot m! \\ &= \frac{(m+n)!}{m+1}. \end{aligned}$$

It follows that

$$\mathbb{E}(X_i) = \Pr(X_i = 1) = \frac{(m+n)!/(m+1)}{(m+n)!} = \frac{1}{m+1},$$

which is the same answer.

The summation $\sum_{i=1}^{n} X_i$ is equal to the total number of beer bottles that are to the left of all cider bottles. If we add one to this, then we get the position of the leftmost cider bottle. Thus,

$$X = 1 + \sum_{i=1}^{n} X_i.$$

This gives

$$\begin{aligned} \mathbb{E}(X) &= \mathbb{E}\left(1 + \sum_{i=1}^{n} X_i\right) \\ &= 1 + \sum_{i=1}^{n} \mathbb{E}(X_i) \\ &= 1 + \sum_{i=1}^{n} \frac{1}{m+1} \\ &= 1 + \frac{n}{m+1} \\ &= \frac{m+n+1}{m+1}. \end{aligned} \tag{3}$$

We now have two expressions for $\mathbb{E}(X)$, one in (2) and the other in (3). These expressions must be equal to each other:

$$m \cdot \sum_{k=1}^{n+1} \frac{\binom{n}{k-1}}{\binom{m+n}{k}} = \frac{m+n+1}{m+1}.$$

By dividing both sides by m, we get

$$\sum_{k=1}^{n+1} \frac{\binom{n}{k-1}}{\binom{m+n}{k}} = \frac{m+n+1}{m(m+1)}.$$

Question 8: You roll a fair die repeatedly, and independently, until you have seen all of the numbers $1, 2, 3, 4, 5, 6$ at least once. Define the random variable

$$X = \text{ the number of times you roll the die.}$$

For example, if you roll the sequence

$$5, 5, 3, 5, 1, 3, 4, 2, 5, 2, 1, 3, 6,$$

then $X = 13$.

Determine the expected value $\mathbb{E}(X)$ of the random variable X.
Hint: Use the Linearity of Expectation. If you have seen exactly i different elements from the set $\{1, 2, 3, 4, 5, 6\}$, how many times do you expect to roll the die until you see a new element from this set?

Solution: The process consists of six stages: For $i = 0, 1, \ldots, 5$,

- Stage i starts at the moment when we have seen exactly i different elements from the set $\{1, 2, \ldots, 6\}$. This stage ends as soon as we see a new element for the first time.

We define X_i to be the number of rolls during stage i. Then

$$X = \sum_{i=0}^{5} X_i$$

and

$$\begin{aligned} \mathbb{E}(X) &= \mathbb{E}\left(\sum_{i=0}^{5} X_i\right) \\ &= \sum_{i=0}^{5} \mathbb{E}(X_i). \end{aligned}$$

Consider stage i: If we roll the die once, then we see a new element with probability $(6-i)/6$. In other words, the success probability is equal to $p_i = (6-i)/6$. We have seen in class that the expected number of rolls until the first success is equal to $1/p_i$. Therefore,

$$\mathbb{E}(X_i) = \frac{1}{p_i} = \frac{6}{6-i}.$$

(This is also true if $i = 0$.) We conclude that

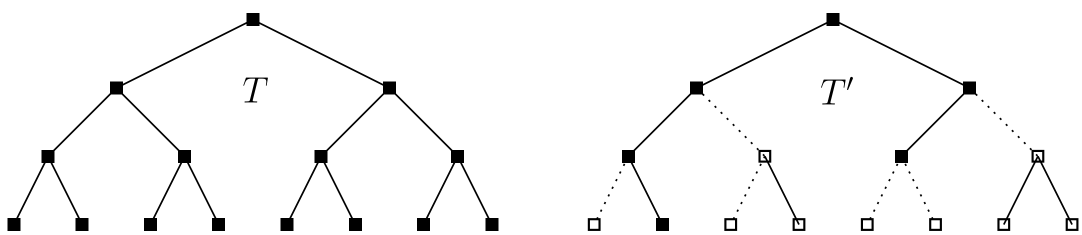
$$\mathbb{E}(X) = \sum_{i=0}^{5} \frac{6}{6-i}.$$

Wolfram Alpha will tell you that this summation is equal to 147/10.

Question 9: Let $k \geq 0$ be an integer and let T be a full binary tree, whose levels are numbered $0, 1, 2, \ldots, k$. (The root is at level 0, whereas the leaves are at level k.) Assume that each edge of T is removed with probability 1/2, independently of other edges. Denote the resulting graph by T'.

Define the random variable X to be the number of nodes that are connected to the root by a path in T'; the root itself is included in X.

In the left figure below, the tree T is shown for the case when $k = 3$. The right figure shows the tree T': The dotted edges are those that have been removed from T, the black nodes are connected to the root by a path in T', whereas the white nodes are not connected to the root by a path in T'. For this case, $X = 6$.



- Let n be the number of nodes in the tree T. Express n in terms of k.
- Prove that the expected value $\mathbb{E}(X)$ of the random variable X is equal to

  $$\mathbb{E}(X) = \log(n+1).$$

  Hint: For any ℓ with $0 \leq \ell \leq k$, how many nodes of T are at level ℓ? Use indicator random variables to determine the expected number of level-ℓ nodes of T that are connected to the root by a path in T'.

Solution: For each $\ell = 0, 1, \ldots, k$, the number of nodes at level ℓ is equal to 2^ℓ. Therefore,

$$n = 1 + 2 + 2^2 + \cdots + 2^k = 2^{k+1} - 1.$$

For each $\ell = 0, 1, \ldots, k$, let X_ℓ be the number of level-ℓ nodes that are connected to the root by a path in T'. Then

$$X = \sum_{\ell=0}^{k} X_\ell$$

and

$$\begin{aligned}\mathbb{E}(X) &= \mathbb{E}\left(\sum_{\ell=0}^{k} X_\ell\right) \\ &= \sum_{\ell=0}^{k} \mathbb{E}\left(X_\ell\right).\end{aligned}$$

Let $0 \leq \ell \leq k$. We are going to determine $\mathbb{E}\left(X_\ell\right)$. We have seen that the number of nodes at level ℓ is equal to 2^ℓ. For each $i = 1, 2, 3, \ldots, 2^\ell$, we define the indicator random variable

$$Y_i = \begin{cases} 1 & \text{if the } i\text{-th node at level } \ell \text{ is connected to the root by a path in } T', \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$X_\ell = \sum_{i=1}^{2^\ell} Y_i$$

and

$$\begin{aligned}\mathbb{E}\left(X_\ell\right) &= \mathbb{E}\left(\sum_{i=1}^{2^\ell} Y_i\right) \\ &= \sum_{i=1}^{2^\ell} \mathbb{E}\left(Y_i\right).\end{aligned}$$

What is $\mathbb{E}\left(Y_i\right)$: Let u be the i-th node at level i. If you walk from u to the root, then you see exactly ℓ many edges. The node u can reach the root in T' if and only all these ℓ edges "survive". Since each edge survives with probability $1/2$, it follows that

$$\mathbb{E}\left(Y_i\right) = \Pr\left(Y_i = 1\right) = (1/2)^\ell.$$

We conclude that

$$\mathbb{E}\left(X_\ell\right) = \sum_{i=1}^{2^\ell} (1/2)^\ell = 1$$

and

$$\mathbb{E}(X) = \sum_{\ell=0}^{k} \mathbb{E}\left(X_\ell\right) = \sum_{\ell=0}^{k} 1 = k + 1.$$

Since $n = 2^{k+1} - 1$, we have $k + 1 = \log(n + 1)$. Thus,

$$\mathbb{E}(X) = \log(n + 1).$$