# COMP 2804 — Assignment 1

**Due:** Thursday January 31, before 11:55pm.

**Assignment Policy:**

- Your assignment must be submitted as one single PDF file through cuLearn.
- **Late assignments will not be accepted. I will not reply to emails of the type "my internet connection broke down at 11:53pm" or "my scanner stopped working at 11:54pm".**
- You are encouraged to collaborate on assignments, but at the level of discussion only. When writing your solutions, you must do so in your own words.
- Past experience has shown conclusively that those who do not put adequate effort into the assignments do not learn the material and have a probability near 1 of doing poorly on the exams.
- When writing your solutions, you must follow the guidelines below.
  - You must justify your answers.
  - The answers should be concise, clear and neat.
  - When presenting proofs, every step should be justified.

**Question 1:**

- Write your name and student number.

**Question 2:** If $n$ and $d$ are positive integers, then $d$ is a *divisor* of $n$, if $n/d$ is an integer.

Determine the number of divisors of the integer

$$1,170,725,783,076,864 = 2^{17} \cdot 3^{12} \cdot 7^5.$$

**Question 3:** Consider permutations of the set $\{a, b, c, d, e, f, g\}$ that do not contain $bge$ (in this order) and do not contain $eaf$ (in this order). Prove that the number of such permutations is equal to 4806.

(You must use counting rules that we have seen in class.)

**Question 4:** Let $n \geq 12$ be an integer and let $\{B_1, B_2, \ldots, B_n\}$ be a set of $n$ beer bottles. Consider permutations of these bottles such that there are exactly 10 bottles between $B_1$ and $B_n$. ($B_1$ can be to the left or right of $B_n$.) Prove that the number of such permutations is equal to

$$\binom{n-2}{10} \cdot 10! \cdot 2 \cdot (n-11)!.$$

**Question 5:** Let $n \geq 3$ be an integer. The *Gn* (or Group of $n$) is an international forum where the $n$ leaders of the world meet to drink beer together. Two of these leaders are Donald Trump and Justin Trudeau. At the end of their meeting, the $n$ leaders stand on a line and a group photo is taken.

- Determine the number of ways in which the $n$ leaders can be arranged on a line, if Donald Trump and Justin Trudeau are standing next to each other.
- Determine the number of ways in which the $n$ leaders can be arranged on a line, if Donald Trump and Justin Trudeau are not standing next to each other.
- Determine the number of ways in which the $n$ leaders can be arranged on a line, if Donald Trump is to the left of Justin Trudeau. (Donald does not necessarily stand immediately to the left of Justin.)

**Question 6:** A *flip* in a bitstring is a pair of adjacent bits that are not equal. For example, the bitstring 010011 has three flips: The first two bits form a flip, the second and third bits form a flip, and the fourth and fifth bits form a flip.

- Determine the number of bitstrings of length 7 that have exactly 3 flips at the following positions: The second and third bits form a flip, the third and fourth bits form a flip, and the fifth and sixth bits form a flip.
- Let $n \geq 2$ and $k$ be integers with $0 \leq k \leq n-1$. Determine the number of bitstrings of length $n$ that have exactly $k$ flips.

**Question 7:** Consider 10 male students $M_1, M_2, \ldots, M_{10}$ and 7 female students $F_1, F_2, \ldots, F_7$. Assume these 17 students are arranged on a horizontal line such that no two female students are standing next to each other. We are interested in the number of such arrangements, where the order of the students matters.

Explain what is wrong with the following argument:

> We are going to use the Product Rule:
>
> - Task 1: Arrange the 7 females on a line. There are 7! ways to do this.
> - Task 2: Choose 6 males. There are $\binom{10}{6}$ ways to do this.
> - Task 3: Place the 6 males chosen in Task 2 in the 6 "gaps" between the females. There are 6! ways to do this.
> - Task 4: At this moment, we have arranged 13 students on a line. We are left with 4 males that have to be placed.
>   - Task 4.1: Place one male. There are 14 ways to do this.
>   - Task 4.2: Place one male. There are 15 ways to do this.
>   - Task 4.3: Place one male. There are 16 ways to do this.
>   - Task 4.4: Place one male. There are 17 ways to do this.
>
> By the Product Rule, the total number of ways to arrange the 17 students is equal to
>
> $$7! \cdot \binom{10}{6} \cdot 6! \cdot 14 \cdot 15 \cdot 16 \cdot 17 = 43,528,181,760,000.$$

**Question 8:** Let $n \geq 2$ be an integer and consider the set $S = \{1, 2, \ldots, n\}$. An ordered triple $(A, x, y)$ is called *awesome*, if (i) $A \subseteq S$, (ii) $x \in A$, and (iii) $y \in A$.

- Let $k$ be an integer with $1 \leq k \leq n$. Determine the number of awesome triples $(A, x, y)$ with $|A| = k$.
- Prove that the number of awesome triples $(A, x, y)$ with $x = y$ is equal to

$$n \cdot 2^{n-1}.$$

- Determine the number of awesome triples $(A, x, y)$ with $x \neq y$.
- Use the above results to prove that

$$\sum_{k=1}^{n} k^2 \binom{n}{k} = n(n-1) \cdot 2^{n-2} + n \cdot 2^{n-1}.$$

**Question 9:** Let $S$ be a set consisting of 19 two-digit integers. Thus, each element of $S$ belongs to the set $\{10, 11, \ldots, 99\}$.

Use the Pigeonhole Principle to prove that this set $S$ contains two distinct elements $x$ and $y$, such that the sum of the two digits of $x$ is equal to the sum of the two digits of $y$.

**Question 10:** Let $S$ be a set consisting of 9 people. Every person $x$ in $S$ has an age $age(x)$, which is an integer with $1 \leq age(x) \leq 60$.

- Assume that there are two people in $S$ having the same age. Prove that there exist two distinct subsets $A$ and $B$ of $S$ such that (i) both $A$ and $B$ are non-empty, (ii) $A \cap B = \emptyset$, and (iii) $\sum_{x \in A} age(x) = \sum_{x \in B} age(x)$.
- Assume that all people in $S$ having different ages. Use the Pigeonhole Principle to prove that there exist two distinct subsets $A$ and $B$ of $S$ such that (i) both $A$ and $B$ are non-empty, and (ii) $\sum_{x \in A} age(x) = \sum_{x \in B} age(x)$.
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