COMP 2804 — Assignment 3

Due: Thursday March 21, before 11:55pm.

Assignment Policy:

- Your assignment must be submitted as one single PDF file through cuLearn.
- **Late assignments will not be accepted. I will not reply to emails of the type "my internet connection broke down at 11:53pm" or "my scanner stopped working at 11:54pm".**
- You are encouraged to collaborate on assignments, but at the level of discussion only. When writing your solutions, you must do so in your own words.
- Past experience has shown conclusively that those who do not put adequate effort into the assignments do not learn the material and have a probability near 1 of doing poorly on the exams.
- When writing your solutions, you must follow the guidelines below.
  - You must justify your answers.
  - The answers should be concise, clear and neat.
  - When presenting proofs, every step should be justified.

Question 1: Write your name and student number.

Question 2: Consider five people, each of which has a uniformly random birthday. (We ignore leap years.) Consider the event

$$A = \text{"at least three people have the same birthday"}.$$

Determine $\Pr(A)$.

Question 3: Consider a box that contains four beer bottles b_1, b_2, b_3, b_4 and two cider bottles c_1, c_2. You choose a uniformly random bottle from the box (and do not put it back), after which you again choose a uniformly random bottle from the box.

Consider the events

$$\begin{aligned} A &= \text{"the first bottle chosen is a beer bottle"}, \\ B &= \text{"the second bottle chosen is a beer bottle"}. \end{aligned}$$

- What is the sample space?
- For each element ω in your sample space, determine $\Pr(\omega)$.

- Determine $\Pr(A)$.
- Determine $\Pr(B)$.
- Are the events A and B independent?

Question 4: A standard deck of 52 cards contains 13 spades (♠), 13 hearts (♡), 13 clubs (♣), and 13 diamonds (♢). You choose a uniformly random card from this deck. Consider the events

$$\begin{aligned} A &= \text{“the chosen card is a clubs or a diamonds card”}, \\ B &= \text{“the chosen card is a clubs or a hearts card”}, \\ C &= \text{“the chosen card is a clubs or a spades card”}. \end{aligned}$$

- Are the events A, B, and C pairwise independent?
- Are the events A, B, and C mutually independent?

Question 5: Consider three events A_1, A_2, and A_3 in some probability space $(S, \Pr)$, and assume that $\Pr(A_1 \cap A_2) > 0$ and $\Pr(A_1) > 0$. Prove that

$$\Pr(A_1 \cap A_2 \cap A_3) = \Pr(A_3 \mid A_1 \cap A_2) \cdot \Pr(A_2 \mid A_1) \cdot \Pr(A_1).$$

Question 6: A standard deck of 52 cards has four Aces.

- You get a uniformly random hand of three cards. Consider the event

  $$A = \text{“the hand consists of three Aces”}.$$

  Determine $\Pr(A)$.
- You get three cards, which are chosen one after another. Each of these three cards is chosen uniformly at random from the current deck of cards. (When a card has been chosen, it is removed from the current deck.) Consider the events

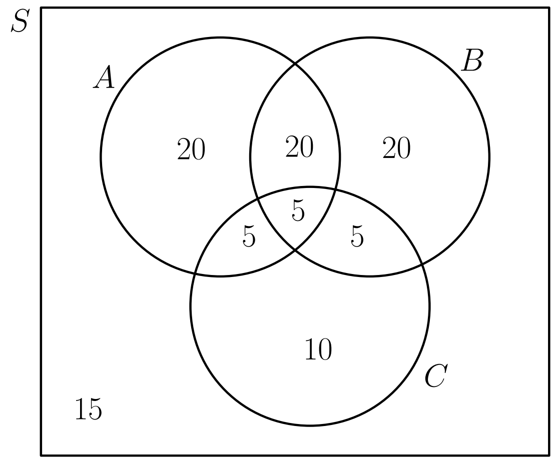
  $$B = \text{“all three cards are Aces”}$$

  and, for $i = 1, 2, 3$,

  $$B_i = \text{“the } i\text{-th card is an Ace.”}$$

  Express the event B in terms of B_1, B_2, and B_3, and use this expression, together with Question 5, to determine $\Pr(B)$.

Question 7: Let S be a sample space consisting of 100 elements. Consider three events A, B, and C as indicated in the figure below. For example, the event A consists of 50 elements, 20 of which are only in A, 20 of which are only in $A \cap B$, 5 of which are only in $A \cap C$, and 5 of which are in $A \cap B \cap C$.



Consider the uniform probability function on this sample space.

- Are the events A and B independent? As always, justify your answer.
- Determine whether or not

$$\Pr(A \cap B \mid C) = \Pr(A \mid C) \cdot \Pr(B \mid C).$$

  Again, justify your answer.

Question 8: Alexa[1] and Zoltan[2] play the following game:

AZ-game:

Step 1: Alexa chooses a uniformly random element from the set $\{1, 2, 3\}$. Let a denote the element that Alexa chooses.
Step 2: Zoltan chooses a uniformly random element from the set $\{1, 2, 3\}$. Let z denote the element that Zoltan chooses.
Step 3: Using one of the three strategies mentioned below, Alexa chooses an element from the set $\{1, 2, 3\} \setminus \{a\}$. Let a' denote the element that Alexa chooses.
Step 4: Using one of the three strategies mentioned below, Zoltan chooses an element from the set $\{1, 2, 3\} \setminus \{z\}$. Let z' denote the element that Alexa chooses.

The AZ-game is a *success* if $a' \neq z'$.

[1] your friendly TA
[2] another friendly TA

- *MinMin Strategy:* In Step 3, Alexa chooses the smallest element in the set $\{1,2,3\} \setminus \{a\}$, and Zoltan chooses the smallest element in the set $\{1,2,3\} \setminus \{z\}$.
  - Describe the sample space for this strategy.
  - For this strategy, determine the probability that the AZ-game is a success.
- *MinMax Strategy:* In Step 3, Alexa chooses the smallest element in the set $\{1,2,3\} \setminus \{a\}$, and Zoltan chooses the largest element in the set $\{1,2,3\} \setminus \{z\}$.
  - Describe the sample space for this strategy.
  - For this strategy, determine the probability that the AZ-game is a success.
- *Random Strategy:* In Step 3, Alexa chooses a uniformly random element in the set $\{1,2,3\} \setminus \{a\}$, and Zoltan chooses a uniformly random element in the set $\{1,2,3\} \setminus \{z\}$.
  - Describe the sample space for this strategy.
  - For this strategy, determine the probability that the AZ-game is a success.

Question 9: You are given a box that contains one red ball and one blue ball. Consider the following algorithm RandomRedBlue(n) that takes as input an integer $n \geq 3$:

```
Algorithm RandomRedBlue(n):

    // n ≥ 3
    // initially, the box contains one red ball and one blue ball
    // all random choices are mutually independent
    for k = 1 to n − 2
    do choose a uniformly random ball in the box;
        if the chosen ball is red
        then put the chosen ball back in the box;
              add one red ball to the box
        else put the chosen ball back in the box;
              add one blue ball to the box
        endif
    endfor
```

For any integers $n \geq 3$ and i with $1 \leq i \leq n-1$, consider the event

$$A_i^n = \text{“at the end of algorithm RandomRedBlue}(n)\text{, the number of red balls in the box is equal to } i\text{”.}$$

In this exercise, you will prove that for any integers $n \geq 3$ and i with $1 \leq i \leq n-1$,

$$\Pr\left(A_i^n\right) = \frac{1}{n-1}. \quad (1)$$

- Let $n \geq 3$ and k be integers with $1 \leq k \leq n-2$. When running algorithm RandomRedBlue(n),
  - how many balls does the box contain at the start of the k-th iteration,
  - how many balls does the box contain at the end of the k-th iteration?
- Let $n \geq 3$ be an integer. After algorithm RandomRedBlue(n) has terminated, how many balls does the box contain?
- For any integer $n \geq 3$, prove that

$$\Pr\left(A_1^n\right) = \frac{1}{n-1}.$$

- For any integer $n \geq 3$, prove that

$$\Pr\left(A_{n-1}^n\right) = \frac{1}{n-1}.$$

- Let $n = 3$. Prove that (1) holds for all values of i in the indicated range.
- Let $n \geq 4$. Consider the event

$$A = \text{"in the } (n-2)\text{-th iteration of algorithm RandomRedBlue}(n)\text{, a red ball is chosen".}$$

  For any integer i with $2 \leq i \leq n-2$, express the event A_i^n in terms of the events A_{i-1}^{n-1}, A_i^{n-1}, and A.
- Let $n \geq 4$. For any integer i with $2 \leq i \leq n-2$, prove that

$$\Pr\left(A_i^n\right) = \Pr\left(A \mid A_{i-1}^{n-1}\right) \cdot \Pr\left(A_{i-1}^{n-1}\right) + \Pr\left(\overline{A} \mid A_i^{n-1}\right) \cdot \Pr\left(A_i^{n-1}\right).$$

- Let $n \geq 4$. Prove that (1) holds for all values of i in the indicated range.