COMP 2804 — Assignment 4

Due: Thursday April 4, before 11:55pm.

Assignment Policy:

- Your assignment must be submitted as one single PDF file through cuLearn.
- **Late assignments will not be accepted. I will not reply to emails of the type "my internet connection broke down at 11:53pm" or "my scanner stopped working at 11:54pm".**
- You are encouraged to collaborate on assignments, but at the level of discussion only. When writing your solutions, you must do so in your own words.
- Past experience has shown conclusively that those who do not put adequate effort into the assignments do not learn the material and have a probability near 1 of doing poorly on the exams.
- When writing your solutions, you must follow the guidelines below.
  - You must justify your answers.
  - The answers should be concise, clear and neat.
  - When presenting proofs, every step should be justified.

Question 1: Write your name and student number.

Question 2: You roll a fair red die and a fair blue die; the two rolls are independent. Let (i, j) be the outcome, where i is the result of the red die and j is the result of the blue die. Consider the random variables

$$X = |i - j|$$

and

$$Y = \max(i, j).$$

Are X and Y independent random variables? Justify your answer.

Question 3: You are given a fair coin.

- You flip this coin twice; the two flips are independent. For each heads, you win 3 dollars, whereas for each tails, you lose 2 dollars. Consider the random variable

$$X = \text{ the amount of money that you win.}$$

  - Use the definition of expected value to determine $\mathbb{E}(X)$.
  - Use the linearity of expectation to determine $\mathbb{E}(X)$.

- You flip this coin 99 times; these flips are mutually independent. For each heads, you win 3 dollars, whereas for each tails, you lose 2 dollars. Consider the random variable

  $$Y = \text{ the amount of money that you win.}$$

  Determine the expected value $\mathbb{E}(Y)$ of Y.

Question 4: You repeatedly flip a fair coin and stop as soon as you get tails followed by heads. (All coin flips are mutually independent.) Consider the random variable

$$X = \text{ the total number of coin flips.}$$

For example, if the sequence of coin flips is $HHHTTTTH$, then $X = 8$.

- Determine the expected value $\mathbb{E}(X)$ of X.

  Hint: Use the linearity of expectation. You may use any result that was proven in class.

Question 5: Let X and Y be two independent random variables on the same sample space. We make the following assumptions:

1. X and Y have the same probability distribution, i.e., for all k,

   $$\Pr(X = k) = \Pr(Y = k).$$

   Note that this implies that $\mathbb{E}(X) = \mathbb{E}(Y)$.

2. For any element in the sample space, the values of both X and Y are non-negative.

Consider the two random variables

$$X' = X^2$$

and

$$Y' = Y^2.$$

- Let a and b be two non-negative real numbers. Prove that

  $$\min\left(a^2, b^2\right) \leq ab.$$

- Prove that

  $$\mathbb{E}\left(\min\left(X', Y'\right)\right) \leq \left(\mathbb{E}(X)\right)^2,$$

  i.e.,

  $$\mathbb{E}\left(\min\left(X^2, Y^2\right)\right) \leq \left(\mathbb{E}(X)\right)^2,$$

  Hint: Since X and Y have the same probability distribution, $\mathbb{E}(X) = \mathbb{E}(Y)$. Since X and Y are independent, $\mathbb{E}(X \cdot Y) = \mathbb{E}(X) \cdot \mathbb{E}(Y)$.

Question 6: Carleton University has implemented a new policy for students who cheat on assignments:

1. When a student is caught cheating, the student meets with the Dean.
2. The Dean has a box that contains n coins. One of these coins has the number n written on it, whereas each of the other $n-1$ coins has the number 1 written on it. Here, n is a very large integer.
3. The student chooses a uniformly random coin from the box.
4. If x is the number written on the chosen coin, then the student gives x^2 bottles of cider to Elisa Kazan.

Consider the random variables

$$\begin{aligned} X &= \text{the number written on the chosen coin,} \\ Z &= \text{the number of bottles of cider that Elisa gets.} \end{aligned}$$

(Note that $Z = X^2$.)

- Prove that
$$\mathbb{E}(X) = 2 - 1/n \leq 2.$$
- Prove that
$$\mathbb{E}(Z) = n + 1 - 1/n \geq n.$$
- Prove that
$$\mathbb{E}\left(X^2\right) \neq O\left((\mathbb{E}(X))^2\right).$$
- By the arguments above, Elisa gets, on average, a very large amount of cider. Since she cannot drink all these bottles, Carleton University changes their policy:
    1. The student chooses a uniformly random coin from the box (and puts it back in the box).
    2. The student chooses a uniformly random coin from the box (and puts it back in the box).
    3. If x is the number written on the first chosen coin, and y is the number written on the second chosen coin, then the student gives $\min(x^2, y^2)$ bottles of cider to Elisa.

  Consider the random variables

$$\begin{aligned} U &= \text{the number written on the first chosen coin,} \\ V &= \text{the number written on the second chosen coin,} \\ W &= \text{the number of bottles of cider that Elisa gets.} \end{aligned}$$

Use Question 5 to prove that

$$\mathbb{E}(W) \leq 4.$$

Question 7: Let $m \geq 1$, $n \geq 1$, and $k \geq 1$ be integers with $k \leq m+n$. Consider a set P consisting of m men and n women. We choose a uniformly random k-element subset Q of P. Consider the random variables

$$\begin{aligned} X &= \text{the number of men in the chosen subset } Q, \\ Y &= \text{the number of women in the chosen subset } Q, \\ Z &= X - Y. \end{aligned}$$

- Prove that
$$\mathbb{E}(Z) = 2 \cdot \mathbb{E}(X) - k.$$
- Determine the expected value $\mathbb{E}(X)$.

  Hint: Denote the men as $M_1, M_2, \ldots, M_m$. Use indicator random variables.
- Prove that
$$\mathbb{E}(Z) = k \cdot \frac{m-n}{m+n}.$$

Question 8: For any integer $n \geq 0$ and any real number x with $0 < x < 1$, define the function

$$F_n(x) = \sum_{k=n}^{\infty} \binom{k}{n} x^k.$$

(Using the *ratio test* from calculus, it can be shown that this infinite series converges for any fixed integer n.)

- Determine a closed form expression for $F_0(x)$. (You may use any result that was proven in class.)
- Let $n \geq 1$ be an integer and let x be a real number with $0 < x < 1$. Prove that
$$F_n(x) = \frac{x}{n} \cdot F_{n-1}(x) + \frac{x^2}{n} \cdot F'_{n-1}(x),$$
  where F'_{n-1} denotes the derivative of F_{n-1}.

  Hint: If $k \geq n \geq 1$, then $\binom{k}{n} = \frac{k}{n}\binom{k-1}{n-1}$.

- Prove that for any integer $n \geq 0$ and any real number x with $0 < x < 1$,

$$F_n(x) = \frac{x^n}{(1-x)^{n+1}},$$

and

$$F'_n(x) = \frac{x^n + n \cdot x^{n-1}}{(1-x)^{n+2}}.$$

- Let $n \geq 0$ and m be integers with $m \geq n+1$. Prove that

$$\sum_{\ell=0}^{\min(n+1,m-n)} (-1)^\ell \binom{n+1}{\ell} \binom{m-\ell}{n} = 0.$$

Hint: You have shown above that

$$(1-x)^{n+1} \sum_{k=n}^{\infty} \binom{k}{n} x^k = (1-x)^{n+1} \cdot F_n(x) = x^n. \tag{1}$$

Use Newton's Binomial Theorem to expand $(1-x)^{n+1}$. Then consider the expansion of the left-hand side in (1). What is the coefficient of x^m in this expansion?

Question 9: Consider a fair red coin and a fair blue coin. We repeatedly flip both coins, and keep track of the number of times that the red coin comes up heads. As soon as the blue coin comes up tails, the process terminates.

A formal description of this process is given in the pseudocode below. The value of the variable i is equal to the number of iterations performed so far, the value of the variable h is equal to the number of times that the red coin came up heads so far, whereas the Boolean variable *stop* is used to decide when the while-loop terminates.

Algorithm RANDOMCOINFLIPS**:**

```
// both the red coin and the blue coin are fair
// all coin flips are mutually independent
i = 0;
h = 0;
stop = false;
while stop = false
do i = i + 1;
    flip the red coin;
    if the result of the red coin is heads
    then h = h + 1
    endif;
    flip the blue coin;
    if the result of the blue coin is tails
    then stop = true
    endif
endwhile;
return i and h
```

Consider the random variables

$$
\begin{aligned}
X &= \text{the value of } i \text{ that is returned by algorithm RandomCoinFlips},\\
Y &= \text{the value of } h \text{ that is returned by algorithm RandomCoinFlips}.
\end{aligned}
$$

Assume that the value of the random variable Y is equal to some integer n. In this exercise, you will determine the expected value of the random variable X.

Thus, we are interested in the *conditional expected value* $\mathbb{E}(X \mid Y = n)$, which is the expected value of X (i.e., the number of iterations of the while-loop), when you are given that the event "$Y = n$" (i.e., during the while-loop, the red coin comes up heads n times) occurs. Formally, we have

$$\mathbb{E}(X \mid Y = n) = \sum_k k \cdot \Pr(X = k \mid Y = n),$$

where the summation ranges over all values of k that X can take.

For the rest of this exercise, let $n \geq 1$ be an integer. The functions F_n and F_n' that are used below are the same as those in Question 8.

- Prove that

$$\Pr(Y = n) = \sum_{k=n}^{\infty} \Pr(Y = n \mid X = k) \cdot \Pr(X = k).$$

- Prove that

$$\Pr(Y = n) = F_n(1/4).$$

- Prove that
$$\mathbb{E}(X \mid Y = n) = \frac{F_n'(1/4)}{4 \cdot F_n(1/4)}.$$
- Prove that
$$\mathbb{E}(X \mid Y = n) = \frac{4n+1}{3}.$$