COMP 2804 — Solutions Assignment 3

Question 1: Write your name and student number.

Solution:

- Name: Johan Cruijff
- Student number: 14

Question 2: Consider five people, each of which has a uniformly random birthday. (We ignore leap years.) Consider the event

$$A = \text{“at least three people have the same birthday”}.$$

Determine $\Pr(A)$.

Solution: We denote the five people by P_1, P_2, P_3, P_4, P_5. The sample space is the set

$$S = \{(b_1, \ldots, b_5) : \text{ each } b_i \in \{1, \ldots, 365\}\}.$$

Here, b_i denotes the birthday of P_i. Note that $|S| = 365^5$.

The event A corresponds to the subset

$$A = \{(b_1, \ldots, b_5) \in S : \text{ at least three birthdays are equal}\}.$$

Since birthdays are uniformly at random, we have

$$\Pr(A) = |A|/|S|.$$

To determine the size of the set A, we consider the different possibilities:

1. All five have the same birthday. There are 365 ways this can happen.
2. Exactly four have the same birthday, and the other person has a different birthday. The number of ways this can happen is
$$\binom{5}{4} \cdot 365 \cdot 364 = 664300.$$
3. Exactly three have the same birthday, and the other two have the same birthday (but different from the three). The number of ways this can happen is
$$\binom{5}{3} \cdot 365 \cdot 364 = 1328600.$$

4. Exactly three have the same birthday, and the other two persons have different birthdays. The number of ways this can happen is

$$\binom{5}{3} \cdot 365 \cdot 364 \cdot 363 = 482281800.$$

Based on this, we have

$$|A| = 365 + 664300 + 1328600 + 482281800 = 484275065$$

and, thus,

$$\Pr(A) = \frac{|A|}{|S|} = \frac{484275065}{365^5} \approx 0.00007475.$$

We can also look at the complement

$$\overline{A} = \{(b_1, \ldots, b_5) \in S : \text{ at most two birthdays are equal}\}.$$

Again, we consider the different possibilities:

1. All five birthdays are different. The number of ways this can happen is

$$365 \cdot 364 \cdot 363 \cdot 362 \cdot 361 = 6302555018760.$$

2. Two people have the same birthday, and the other three have different birthdays. The number of ways this can happen is

$$\binom{5}{2} \cdot 365 \cdot 364 \cdot 363 \cdot 362 = 174586011600.$$

3. One pair has the same birthday, one other pair has the same birthday (but different from the first pair), and the other person has a different birthday.

   For this case, we have to be careful to avoid double counting:

   (a) Choose a subset of size four: There are $\binom{5}{4}$ ways to do this.

   (b) Divide the chosen subset into two pairs (the people in the same pair will get the same birthday). We do this as follows: Sort the chosen subset by name. For example, if the chosen subset consists of Zoltan, Alexa, Michiel, and Julia, then we get the ordered sequence

   Alexa, Julia, Michiel, Zoltan.

   Put Alexa in one pair. There are 3 ways to do this. Once this is done, the remaining two people form one pair.

(c) Choose a birthday for the first pair, then choose a different birthday for the second pair, and finally choose a birthday for the remaining person. There are $365 \cdot 364 \cdot 363$ ways to do this.

Conclusion: the number of ways this third case can happen is

$$\binom{5}{4} \cdot 3 \cdot 365 \cdot 364 \cdot 363 = 723422700.$$

Based on this, we have

$$|\overline{A}| = 6302555018760 + 174586011600 + 723422700 = 6477864453060.$$

This gives

$$\Pr(A) = 1 - \Pr\left(\overline{A}\right) = 1 - \frac{6477864453060}{365^5} = 0.00007475.$$

Question 3: Consider a box that contains four beer bottles b_1, b_2, b_3, b_4 and two cider bottles c_1, c_2. You choose a uniformly random bottle from the box (and do not put it back), after which you again choose a uniformly random bottle from the box.

Consider the events

$$\begin{aligned} A &= \text{“the first bottle chosen is a beer bottle”}, \\ B &= \text{“the second bottle chosen is a beer bottle”}. \end{aligned}$$

- What is the sample space?
- For each element ω in your sample space, determine $\Pr(\omega)$.
- Determine $\Pr(A)$.
- Determine $\Pr(B)$.
- Are the events A and B independent?

Solution: Let

$$X = \{b_1, b_2, b_3, b_4, c_1, c_2\}.$$

Then the sample space is the set

$$S = \{(x, y) : x \in X, y \in X, x \neq y\}.$$

Here, x is the first bottle that is chosen, and y is the second bottle. Note that $|S| = 6 \cdot 5 = 30$.

Consider one element (x, y) in S. We have to determine $\Pr(x, y)$. The probability of choosing x is $1/6$. After x has been chosen, we choose y with probability $1/5$. Therefore, $\Pr(x, y) = 1/6 \cdot 1/3 = 1/30$. Thus, we have a uniform distribution.

Here is a more formal argument to do this: Consider the events

$$A_x = \text{ "the first bottle chosen is } x\text{"}$$

and

$$B_y = \text{ "the second bottle chosen is } y\text{".}$$

Then

$$\Pr(x,y) = \Pr(A_x \wedge B_y) = \Pr(B_y \mid A_x) \cdot \Pr(A_x) = 1/5 \cdot 1/6 = 1/30.$$

Next, we determine $\Pr(A)$: Note that

$$A = \{(x,y) \in S : x = \text{ beer, } y \neq x\}.$$

There are 4 choices for x and 5 choices for y. Thus, $|A| = 4 \cdot 5 = 20$ and we get

$$\Pr(A) = \frac{|A|}{|S|} = \frac{20}{30} = 2/3.$$

Next, we determine $\Pr(B)$: Note that

$$B = \{(x,y) \in S : y = \text{ beer, } x \neq y\}.$$

There are 4 choices for y and 5 choices for x. Thus, $|B| = 4 \cdot 5 = 20$ and we get

$$\Pr(B) = \frac{|B|}{|S|} = \frac{20}{30} = 2/3.$$

To determine if A and B are independent events, we compute $\Pr(A \cap B)$: Note that

$$A \cap B = \{(x,y) \in S : x = \text{ beer, } y = \text{ beer, } x \neq y\}.$$

There are 4 choices for x and 3 choices for y. Thus, $|A \cap B| = 4 \cdot 3 = 12$ and we get

$$\Pr(A \cap B) = \frac{|A \cap B|}{|S|} = \frac{12}{30} = 2/5.$$

Since

$$\Pr(A \cap B) \neq \Pr(A) \cdot \Pr(B),$$

the events A and B are not independent.

Question 4: A standard deck of 52 cards contains 13 spades ($\spadesuit$), 13 hearts ($\heartsuit$), 13 clubs ($\clubsuit$), and 13 diamonds ($\diamondsuit$). You choose a uniformly random card from this deck. Consider the events

$$\begin{aligned} A &= \text{ "the chosen card is a clubs or a diamonds card",} \\ B &= \text{ "the chosen card is a clubs or a hearts card",} \\ C &= \text{ "the chosen card is a clubs or a spades card".} \end{aligned}$$

- Are the events A, B, and C pairwise independent?
- Are the events A, B, and C mutually independent?

Solution: The sample space is the set S consisting of the 52 cards in the deck.

Since there are 13 clubs and 13 diamonds, we have

$$\Pr(A) = \frac{13+13}{52} = 1/2.$$

By the same reasoning, we have

$$\Pr(B) = \Pr(C) = \frac{13+13}{52} = 1/2.$$

The event $A \cap B$ corresponds to chosing a clubs. Since there are 13 clubs, we have

$$\Pr(A \cap B) = \frac{13}{52} = 1/4.$$

By the same reasoning, we have

$$\Pr(A \cap C) = \Pr(B \cap C) = \frac{13}{52} = 1/4.$$

We conclude that

$$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B),$$
$$\Pr(A \cap C) = \Pr(A) \cdot \Pr(C),$$
$$\Pr(B \cap C) = \Pr(B) \cdot \Pr(C).$$

Thus, the events A, B, and C are pairwise independent.

The event $A \cap B \cap C$ corresponds to chosing a clubs. Since there are 13 clubs, we have

$$\Pr(A \cap B \cap C) = \frac{13}{52} = 1/4.$$

Thus,

$$\Pr(A \cap B \cap C) \neq \Pr(A) \cdot \Pr(B) \cdot \Pr(C).$$

Thus, the events A, B, and C are not mutually independent.

Question 5: Consider three events A_1, A_2, and A_3 in some probability space $(S, \Pr)$, and assume that $\Pr(A_1 \cap A_2) > 0$ and $\Pr(A_1) > 0$. Prove that

$$\Pr(A_1 \cap A_2 \cap A_3) = \Pr(A_3 \mid A_1 \cap A_2) \cdot \Pr(A_2 \mid A_1) \cdot \Pr(A_1).$$

Solution: The claim follows by using the definition of conditional probability:

$$\begin{aligned}
&\Pr(A_3 \mid A_1 \cap A_2) \cdot \Pr(A_2 \mid A_1) \cdot \Pr(A_1) \\
&= \frac{\Pr(A_1 \cap A_2 \cap A_3)}{\Pr(A_1 \cap A_2)} \cdot \frac{\Pr(A_1 \cap A_2)}{\Pr(A_1)} \cdot \Pr(A_1) \\
&= \Pr(A_1 \cap A_2 \cap A_3).
\end{aligned}$$

Question 6: A standard deck of 52 cards has four Aces.

- You get a uniformly random hand of three cards. Consider the event

  $$A = \text{“the hand consists of three Aces”}.$$

  Determine $\Pr(A)$.

  Solution: The sample space is the set S consisting of all hands (i.e., subsets) of three cards. Note that $|S| = \binom{52}{3}$.

  Since there are 4 Aces, the number of hands of three cards that are all Aces is equal to $\binom{4}{3} = 4$. Thus,

  $$\Pr(A) = \frac{|A|}{|S|} = \frac{4}{\binom{52}{3}} = \frac{1}{13 \cdot 17 \cdot 25}.$$

- You get three cards, which are chosen one after another. Each of these three cards is chosen uniformly at random from the current deck of cards. (When a card has been chosen, it is removed from the current deck.) Consider the events

  $$B = \text{“all three cards are Aces”}$$

  and, for $i = 1, 2, 3$,

  $$B_i = \text{“the } i\text{-th card is an Ace.”}$$

  Express the event B in terms of B_1, B_2, and B_3, and use this expression, together with Question 5, to determine $\Pr(B)$.

  Solution: It should be obvious that

  $$B = B_1 \cap B_2 \cap B_3.$$

  Using Question 5, we get

  $$\begin{aligned}
  \Pr(B) &= \Pr(B_1 \cap B_2 \cap B_3) \\
  &= \Pr(B_3 \mid B_1 \cap B_2) \cdot \Pr(B_2 \mid B_1) \cdot \Pr(B_1).
  \end{aligned}$$

  We determine the terms on the right-hand side:

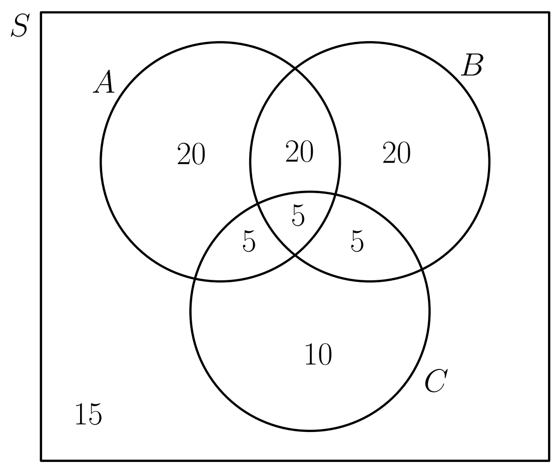
  1. It is clear that

     $$\Pr(B_1) = \frac{4}{52} = \frac{1}{13}.$$

2. To determine $\Pr(B_2 \mid B_1)$, we assume that the event B_1 occurs. Thus, the first card is an Ace. For the second card, there are 3 Aces left, out of 51 cards. This gives
$$\Pr(B_2 \mid B_1) = \frac{3}{51} = \frac{1}{17}.$$
3. To determine $\Pr(B_3 \mid B_1 \cap B_2)$, we assume that both the events B_1 and B_2 occur. Thus, the first two cards are Aces. For the third card, there are 2 Aces left, out of 50 cards. This gives
$$\Pr(B_3 \mid B_1 \cap B_2) = \frac{2}{50} = \frac{1}{25}.$$

We conclude that
$$\Pr(B) = \frac{1}{25} \cdot \frac{1}{17} \cdot \frac{1}{13} = \frac{1}{13 \cdot 17 \cdot 25}.$$

Question 7: Let S be a sample space consisting of 100 elements. Consider three events A, B, and C as indicated in the figure below. For example, the event A consists of 50 elements, 20 of which are only in A, 20 of which are only in $A \cap B$, 5 of which are only in $A \cap C$, and 5 of which are in $A \cap B \cap C$.



Consider the uniform probability function on this sample space.

- Are the events A and B independent? As always, justify your answer.
- Determine whether or not
$$\Pr(A \cap B \mid C) = \Pr(A \mid C) \cdot \Pr(B \mid C).$$
Again, justify your answer.

Solution: From the Venn diagram, we see that $|A| = 50$, $|B| = 50$, and $|A \cap B| = 25$. Therefore,

$$\Pr(A) = \frac{50}{100} = \frac{1}{2},$$
$$\Pr(B) = \frac{50}{100} = \frac{1}{2},$$

and

$$\Pr(A \cap B) = \frac{25}{100} = \frac{1}{4}.$$

Since

$$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B),$$

the events A and B are independent.

Next, we determine the conditional probabilities: From the Venn diagram, we see that $|C| = 25$, $|A \cap C| = 10$, $|B \cap C| = 10$, and $|A \cap B \cap C| = 5$. Therefore,

$$\Pr(A \mid C) = \frac{\Pr(A \cap C)}{\Pr(C)} = \frac{10/100}{25/100} = \frac{2}{5},$$

$$\Pr(B \mid C) = \frac{\Pr(B \cap C)}{\Pr(C)} = \frac{10/100}{25/100} = \frac{2}{5},$$

and

$$\Pr(A \cap B \mid C) = \frac{\Pr(A \cap B \cap C)}{\Pr(C)} = \frac{5/100}{25/100} = \frac{1}{5}.$$

We conclude that

$$\Pr(A \cap B \mid C) \neq \Pr(A \mid C) \cdot \Pr(B \mid C).$$

Remark: Even though A and B are independent, the "multiplication rule" for independent events does not apply to conditional probabilities.

Question 8: Alexa[1] and Zoltan[2] play the following game:

AZ-game:

Step 1: Alexa chooses a uniformly random element from the set $\{1, 2, 3\}$. Let a denote the element that Alexa chooses.
Step 2: Zoltan chooses a uniformly random element from the set $\{1, 2, 3\}$. Let z denote the element that Zoltan chooses.
Step 3: Using one of the three strategies mentioned below, Alexa chooses an element from the set $\{1, 2, 3\} \setminus \{a\}$. Let a' denote the element that Alexa chooses.
Step 4: Using one of the three strategies mentioned below, Zoltan chooses an element from the set $\{1, 2, 3\} \setminus \{z\}$. Let z' denote the element that Alexa chooses.

The AZ-game is a *success* if $a' \neq z'$.

[1] your friendly TA
[2] another friendly TA

- *MinMin Strategy:* In Step 3, Alexa chooses the smallest element in the set $\{1, 2, 3\} \setminus \{a\}$, and Zoltan chooses the smallest element in the set $\{1, 2, 3\} \setminus \{z\}$.
  - Describe the sample space for this strategy.
  - For this strategy, determine the probability that the AZ-game is a success.
- *MinMax Strategy:* In Step 3, Alexa chooses the smallest element in the set $\{1, 2, 3\} \setminus \{a\}$, and Zoltan chooses the largest element in the set $\{1, 2, 3\} \setminus \{z\}$.
  - Describe the sample space for this strategy.
  - For this strategy, determine the probability that the AZ-game is a success.
- *Random Strategy:* In Step 3, Alexa chooses a uniformly random element in the set $\{1, 2, 3\} \setminus \{a\}$, and Zoltan chooses a uniformly random element in the set $\{1, 2, 3\} \setminus \{z\}$.
  - Describe the sample space for this strategy.
  - For this strategy, determine the probability that the AZ-game is a success.

Solution: We start with the MinMin Strategy: The only places where random choices are made is in Steps 1 and 2. Therefore, the sample space is the set

$$S = \{(a, z) : a, z \in \{1, 2, 3\}\}.$$

Note that $|S| = 3 \cdot 3 = 9$ and we have a uniform probability.

The following table gives all possibilities for a, z, a', and z', and whether or not the game is a success (s) or failure (f):

a	z	a'	z'	s/f
1	1	2	2	f
1	2	2	1	s
1	3	2	1	s
2	1	1	2	s
2	2	1	1	f
2	3	1	1	f
3	1	1	2	s
3	2	1	1	f
3	3	1	1	f

We see that, out of the 9 possibilities, 4 are successful. Therefore, the probability that the AZ-game is a success is equal to 4/9.

Next we do the MinMax Strategy: The only places where random choices are made is in Steps 1 and 2. Therefore, the sample space is the set

$$S = \{(a, z) : a, z \in \{1, 2, 3\}\}.$$

Note that $|S| = 3 \cdot 3 = 9$ and we have a uniform probability.

The following table gives all possibilities for a, z, a', and z', and whether or not the game is a success (s) or failure (f):

a	z	a'	z'	s/f
1	1	2	3	s
1	2	2	3	s
1	3	2	2	f
2	1	1	3	s
2	2	1	3	s
2	3	1	2	s
3	1	1	3	s
3	2	1	3	s
3	3	1	2	s

We see that, out of the 9 possibilities, 8 are successful. Therefore, the probability that the AZ-game is a success is equal to 8/9.

Finally we do the Random Strategy: Since a random choice is made in each of the four steps, the sample space is the set

$$S = \{(a, a', z, z') : a, a', z, z' \in \{1, 2, 3\}, a' \neq a, z' \neq z\}.$$

Note that $|S| = 3 \cdot 2 \cdot 3 \cdot 2 = 36$ and we have a uniform probability.

How many elements in the sample space lead to a success:

1. Consider elements (a, a', z, z') for which $z = a'$ and $z' \neq a'$.

   There are 3 choices for a, 2 choices for a', 1 choice for z, and 2 choices for z'.

   Thus, the number of elements for this case is $3 \cdot 2 \cdot 1 \cdot 2 = 12$.

2. Consider elements (a, a', z, z') for which $z \neq a'$ and $z' \neq a'$.

   There are 3 choices for a, 2 choices for a', 2 choices for z, and 1 choice for z'.

   Thus, the number of elements for this case is $3 \cdot 2 \cdot 2 \cdot 1 = 12$.

We see that, out of the 36 possibilities, $12 + 12 = 24$ are successful. Therefore, the probability that the AZ-game is a success is equal to $24/36 = 2/3$.

Question 9: You are given a box that contains one red ball and one blue ball. Consider the following algorithm RandomRedBlue(n) that takes as input an integer $n \geq 3$:

```
Algorithm RandomRedBlue(n):

    // n ≥ 3
    // initially, the box contains one red ball and one blue ball
    // all random choices are mutually independent
    for k = 1 to n − 2
    do choose a uniformly random ball in the box;
        if the chosen ball is red
        then put the chosen ball back in the box;
              add one red ball to the box
        else put the chosen ball back in the box;
             add one blue ball to the box
        endif
    endfor
```

For any integers $n \geq 3$ and i with $1 \leq i \leq n-1$, consider the event

$$A_i^n = \text{"at the end of algorithm RandomRedBlue}(n)\text{, the number of red balls in the box is equal to } i\text{"}.$$

In this exercise, you will prove that for any integers $n \geq 3$ and i with $1 \leq i \leq n-1$,

$$\Pr\left(A_i^n\right) = \frac{1}{n-1}. \tag{1}$$

- Let $n \geq 3$ and k be integers with $1 \leq k \leq n-2$. When running algorithm RandomRedBlue(n),
  - how many balls does the box contain at the start of the k-th iteration,
  - how many balls does the box contain at the end of the k-th iteration?
- Let $n \geq 3$ be an integer. After algorithm RandomRedBlue(n) has terminated, how many balls does the box contain?
- For any integer $n \geq 3$, prove that

$$\Pr\left(A_1^n\right) = \frac{1}{n-1}.$$

- For any integer $n \geq 3$, prove that

$$\Pr\left(A_{n-1}^n\right) = \frac{1}{n-1}.$$

- Let $n = 3$. Prove that (1) holds for all values of i in the indicated range.
- Let $n \geq 4$. Consider the event

$$A = \text{"in the } (n-2)\text{-th iteration of algorithm RandomRedBlue}(n), \text{ a red ball is chosen".}$$

  For any integer i with $2 \leq i \leq n-2$, express the event A_i^n in terms of the events A_{i-1}^{n-1}, A_i^{n-1}, and A.
- Let $n \geq 4$. For any integer i with $2 \leq i \leq n-2$, prove that

$$\Pr\left(A_i^n\right) = \Pr\left(A \mid A_{i-1}^{n-1}\right) \cdot \Pr\left(A_{i-1}^{n-1}\right) + \Pr\left(\overline{A} \mid A_i^{n-1}\right) \cdot \Pr\left(A_i^{n-1}\right).$$

- Let $n \geq 4$. Prove that (1) holds for all values of i in the indicated range.

Solution:

First part:

- At the start of the 1-st iteration, the box contains 2 balls.
- In one iteration, the number of balls increases by 1.
- From this, it follows that
  - at the start of the k-th iteration, the box contains $k+1$ balls,
  - at the end of the k-th iteration, the box contains $k+2$ balls.

Second part: When algorithm RandomRedBlue(n) terminates, we are at the end of the $(n-2)$-th iteration. From the previous part, the box contains n balls.

Third part: the event A_1^n says that at the end of algorithm RandomRedBlue(n), the box contains exactly 1 red ball. This means that in each iteration, a blue ball is chosen (and one blue ball is added to the box).

Consider the k-th iteration. At the start of this iteration, the box contains $k+1$ balls. Assuming that so far, we always chose a blue ball, this means that the box contains 1 red ball and k blue balls. The probability that we choose a blue ball in the k-th iteration is equal to $k/(k+1)$.

Since this must happen for all $k = 1, 2, \ldots, n-2$, we get

$$\begin{aligned}
\Pr\left(A_1^n\right) &= \prod_{k=1}^{n-2} \frac{k}{k+1} \\
&= \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} \cdots \frac{n-2}{n-1} \\
&= \frac{1}{n-1}.
\end{aligned}$$

Fourth part: the event A^n_{n-1} says that at the end of algorithm RandomRedBlue(n), the box contains exactly $n-1$ red balls. This means that in each iteration, a red ball is chosen (and one red ball is added to the box). By exactly the same computation as for the third part, we get

$$\Pr\left(A^n_{n-1}\right) = \frac{1}{n-1}.$$

Fifth part: Let $n=3$. We have to show that (1) holds for all values of i in the indicated range. Thus, we copy (1) and replace n by 3: We have to prove that for all i with $1 \leq i \leq 2$,

$$\Pr\left(A^3_i\right) = \frac{1}{2}.$$

We just proved this in the third and fourth parts.

Sixth part: We first oberve that

RandomRedBlue(n) is the same as RandomRedBlue($n-1$) plus the $(n-2)$-th iteration.

The event A^n_i says that after RandomRedBlue(n) has terminated, the box contains i red balls. There are two possibilities for this to happen:

1. After RandomRedBlue($n-1$) has terminated, the box contains $i-1$ red balls and a red ball is chosen in the $(n-2)$-th iteration.
2. After RandomRedBlue($n-1$) has terminated, the box contains i red balls and a blue ball is chosen in the $(n-2)$-th iteration.

Based on this, we get

$$A^n_i \Longleftrightarrow \left(A^{n-1}_{i-1} \wedge A\right) \vee \left(A^{n-1}_i \wedge \overline{A}\right).$$

Seventh part: Using the previous part, the fact that the two events $A^{n-1}_{i-1} \wedge A$ and $A^{n-1}_i \wedge \overline{A}$ are disjoint, and the definition of conditional probability, we get

$$\begin{aligned}
\Pr\left(A^n_i\right) &= \Pr\left(\left(A^{n-1}_{i-1} \wedge A\right) \vee \left(A^{n-1}_i \wedge \overline{A}\right)\right) \\
&= \Pr\left(A^{n-1}_{i-1} \wedge A\right) + \Pr\left(A^{n-1}_i \wedge \overline{A}\right) \\
&= \Pr\left(A \mid A^{n-1}_{i-1}\right) \cdot \Pr\left(A^{n-1}_{i-1}\right) + \Pr\left(\overline{A} \mid A^{n-1}_i\right) \cdot \Pr\left(A^{n-1}_i\right).
\end{aligned}$$

Eighth part: The proof is by induction on n. We have already proved that (1) holds for $n=3$. Let $n \geq 4$, and assume that (1) is true for $n-1$. Let i be an integer with $1 \leq i \leq n-1$. We are going to prove that (1) holds for i and n.

In the third and fourth parts, we have shown that (1) holds for cases when $i=1$ and $i=n$.

From now on, we assume that $2 \leq i \leq n-2$. We have seen above that

$$\Pr\left(A^n_i\right) = \Pr\left(A \mid A^{n-1}_{i-1}\right) \cdot \Pr\left(A^{n-1}_{i-1}\right) + \Pr\left(\overline{A} \mid A^{n-1}_i\right) \cdot \Pr\left(A^{n-1}_i\right).$$

We have to show that the left-hand side is equal to $1/(n-1)$. We do this by determining the four terms on the right-hand side:

1. $\Pr\left(A_{i-1}^{n-1}\right) = 1/(n-2)$; this follows by induction.

2. $\Pr\left(A_{i}^{n-1}\right) = 1/(n-2)$; this follows by induction.

3. What is $\Pr\left(A \mid A_{i-1}^{n-1}\right)$?

   We are given that the event A_{i-1}^{n-1} occurs. Thus, at the end of the $(n-3)$-th iteration, the box contains $n-1$ balls, $i-1$ of which are red. Given this, the event A says that we choose a red ball. Thus,

$$\Pr\left(A \mid A_{i-1}^{n-1}\right) = \frac{i-1}{n-1}.$$

4. What is $\Pr\left(\overline{A} \mid A_{i}^{n-1}\right)$?

   We are given that the event A_{i}^{n-1} occurs. Thus, at the end of the $(n-3)$-th iteration, the box contains $n-1$ balls, i of which are red. Given this, the event $\overline{A}$ says that we choose a blue ball. Thus,

$$\Pr\left(\overline{A} \mid A_{i}^{n-1}\right) = \frac{n-1-i}{n-1}.$$

By putting everything together, we get

$$\begin{aligned}
\Pr\left(A_i^n\right) &= \frac{i-1}{n-1} \cdot \frac{1}{n-2} + \frac{n-1-i}{n-1} \cdot \frac{1}{n-2} \\
&= \frac{n-2}{(n-1)(n-2)} \\
&= \frac{1}{n-1}.
\end{aligned}$$

This was awesome eh?