COMP 2804 — Solutions Assignment 4

Question 1: Write your name and student number.

Solution:

- Name: Lionel Messi
- Student number: 10

Question 2: You roll a fair red die and a fair blue die; the two rolls are independent. Let (i, j) be the outcome, where i is the result of the red die and j is the result of the blue die. Consider the random variables

$$X = |i - j|$$

and

$$Y = \max(i, j).$$

Are X and Y independent random variables? Justify your answer.

Solution: To answer this question, we try to discover some dependencies between X and Y: If $Y = 1$, then both i and j must be equal to 1 and, thus, X must be 0. This suggests that X and Y are not independent. To formally prove this, consider the equation

$$\Pr(X = 1 \wedge Y = 1) \stackrel{?}{=} \Pr(X = 1) \cdot \Pr(Y = 1).$$

1. The left-hand side is equal to 0, because the event "$X = 1 \wedge Y = 1$" cannot happen.
2. $\Pr(X = 1) \neq 0$, because the event $X = 1$ can happen.
3. $\Pr(Y = 1) \neq 0$, because the event $Y = 1$ can happen.
4. Thus, the right-hand side is non-zero.

We conclude that

$$\Pr(X = 1 \wedge Y = 1) \neq \Pr(X = 1) \cdot \Pr(Y = 1)$$

and X and Y are not independent.

Another example that works: If $X = 5$, then $\{i, j\} = \{1, 6\}$ and, thus, $Y = 6$.

Question 3: You are given a fair coin.

- You flip this coin twice; the two flips are independent. For each heads, you win 3 dollars, whereas for each tails, you lose 2 dollars. Consider the random variable

$$X = \text{ the amount of money that you win.}$$

– Use the definition of expected value to determine $\mathbb{E}(X)$.

Solution: The sample space is $S = \{HH, HT, TH, TT\}$. If we express the random variable X as a function, we get

$$X(HH) = 6, X(HT) = 1, X(TH) = 1, X(TT) = -4.$$

By the definition of expected value, we get

$$\begin{aligned}
\mathbb{E}(X) &= \sum_{\omega \in S} X(\omega) \cdot \Pr(\omega) \\
&= X(HH) \cdot \Pr(HH) + X(HT) \cdot \Pr(HT) + \\
&\quad X(TH) \cdot \Pr(TH) + X(TT) \cdot \Pr(TT) \\
&= 6 \cdot 1/4 + 1 \cdot 1/4 + 1 \cdot 1/4 - 4 \cdot 1/4 \\
&= 1.
\end{aligned}$$

– Use the linearity of expectation to determine $\mathbb{E}(X)$.

Solution: For $i = 1, 2$, let X_i be the amount that we win on the i-th flip. Then

$$X = X_1 + X_2,$$

$$\mathbb{E}(X_1) = 3 \cdot 1/2 - 2 \cdot 1/2 = 1/2,$$

and

$$\mathbb{E}(X_2) = 3 \cdot 1/2 - 2 \cdot 1/2 = 1/2.$$

By the linearity of expectation, we get

$$\mathbb{E}(X) = \mathbb{E}(X_1 + X_2) = \mathbb{E}(X_1) + \mathbb{E}(X_2) = 1/2 + 1/2 = 1.$$

- You flip this coin 99 times; these flips are mutually independent. For each heads, you win 3 dollars, whereas for each tails, you lose 2 dollars. Consider the random variable

$$Y = \text{ the amount of money that you win.}$$

Determine the expected value $\mathbb{E}(Y)$ of Y.

Solution: For $i = 1, 2, \ldots, 99$, let Y_i be the amount that we win on the i-th flip. Then

$$Y = \sum_{i=1}^{99} Y_i$$

and

$$\mathbb{E}(Y_i) = 3 \cdot 1/2 - 2 \cdot 1/2 = 1/2.$$

Using the linearity of expectation, we get

$$\mathbb{E}(Y) = \mathbb{E}\left(\sum_{i=1}^{99} Y_i\right) = \sum_{i=1}^{99} \mathbb{E}(Y_i) = \sum_{i=1}^{99} 1/2 = 99/2.$$

Question 4: You repeatedly flip a fair coin and stop as soon as you get tails followed by heads. (All coin flips are mutually independent.) Consider the random variable

$$X = \text{ the total number of coin flips.}$$

For example, if the sequence of coin flips is $HHHTTTTH$, then $X = 8$.

- Determine the expected value $\mathbb{E}(X)$ of X.

  Hint: Use the linearity of expectation. You may use any result that was proven in class.

Solution: Each possible sequence of coin flips is of the form $H^n T T^m H$, for some integers $n \geq 0$ and $m \geq 0$. We divide each such sequence into two phases:

- Phase 1: The flips up to, and including, the first T. These are sequences of the form $H^n T$, for some $n \geq 0$.
- Phase 2: The flips after Phase 1. These are sequences of the form $T^m H$, for some $m \geq 0$.

For $i = 1, 2$, let X_i be the number of coin flips in Phase i. Then

$$X = X_1 + X_2.$$

In class, we have seen the following: If an experiment has success probability p, and we repeat the experiment until we are successful for the first time, then the expected number of times we do the experiment is equal to $1/p$.

If we take $p = 1/2$, then we see that

$$\mathbb{E}\left(X_1\right) = \frac{1}{1/2} = 2$$

and

$$\mathbb{E}\left(X_2\right) = \frac{1}{1/2} = 2.$$

This gives

$$\mathbb{E}(X) = \mathbb{E}\left(X_1 + X_2\right) = \mathbb{E}\left(X_1\right) + \mathbb{E}\left(X_2\right) = 2 + 2 = 4.$$

Question 5: Let X and Y be two independent random variables on the same sample space. We make the following assumptions:

1. X and Y have the same probability distribution, i.e., for all k,

   $$\Pr(X = k) = \Pr(Y = k).$$

   Note that this implies that $\mathbb{E}(X) = \mathbb{E}(Y)$.

2. For any element in the sample space, the values of both X and Y are non-negative.

Consider the two random variables

$$X' = X^2$$

and

$$Y' = Y^2.$$

- Let a and b be two non-negative real numbers. Prove that

$$\min\left(a^2, b^2\right) \leq ab.$$

  Solution: If $a \leq b$, then

$$\min\left(a^2, b^2\right) = a^2 = a \cdot a \leq ab.$$

  (Note that we use the fact that $a \geq 0$ and $b \geq 0$.)
  If $b \leq a$, then

$$\min\left(a^2, b^2\right) = b^2 = b \cdot b \leq ab.$$

- Prove that

$$\mathbb{E}\left(\min\left(X', Y'\right)\right) \leq \left(\mathbb{E}(X)\right)^2,$$

  i.e.,

$$\mathbb{E}\left(\min\left(X^2, Y^2\right)\right) \leq \left(\mathbb{E}(X)\right)^2,$$

  Hint: Since X and Y have the same probability distribution, $\mathbb{E}(X) = \mathbb{E}(Y)$. Since X and Y are independent, $\mathbb{E}(X \cdot Y) = \mathbb{E}(X) \cdot \mathbb{E}(Y)$.

  Solution: From the first part, we get

$$\min\left(X^2, Y^2\right) \leq XY,$$

  implying that

$$\mathbb{E}\left(\min\left(X^2, Y^2\right)\right) \leq \mathbb{E}(XY).$$

  From the hint,

$$\mathbb{E}(XY) = \mathbb{E}(X) \cdot \mathbb{E}(Y) = \mathbb{E}(X) \cdot \mathbb{E}(X) = \left(\mathbb{E}(X)\right)^2.$$

  We conclude that

$$\mathbb{E}\left(\min\left(X^2, Y^2\right)\right) \leq \left(\mathbb{E}(X)\right)^2.$$

Question 6: Carleton University has implemented a new policy for students who cheat on assignments:

1. When a student is caught cheating, the student meets with the Dean.

2. The Dean has a box that contains n coins. One of these coins has the number n written on it, whereas each of the other $n-1$ coins has the number 1 written on it. Here, n is a very large integer.

3. The student chooses a uniformly random coin from the box.

4. If x is the number written on the chosen coin, then the student gives x^2 bottles of cider to Elisa Kazan.

Consider the random variables

$$\begin{aligned} X &= \text{the number written on the chosen coin,} \\ Z &= \text{the number of bottles of cider that Elisa gets.} \end{aligned}$$

(Note that $Z = X^2$.)

- Prove that
$$\mathbb{E}(X) = 2 - 1/n \leq 2.$$

  Solution: The possible values for X are 1 (with probability $(n-1)/n$) and n (with probability $1/n$). It follows that

$$\begin{aligned} \mathbb{E}(X) &= 1 \cdot \Pr(X=1) + n \cdot \Pr(X=n) \\ &= 1 \cdot \frac{n-1}{n} + n \cdot \frac{1}{n} \\ &= 2 - \frac{1}{n} \\ &\leq 2. \end{aligned}$$

- Prove that
$$\mathbb{E}(Z) = n + 1 - 1/n \geq n.$$

  Solution: The possible values for Z are 1 (with probability $(n-1)/n$) and n^2 (with probability $1/n$). It follows that

$$\begin{aligned} \mathbb{E}(Z) &= 1 \cdot \Pr(Z=1) + n^2 \cdot \Pr\left(Z=n^2\right) \\ &= 1 \cdot \frac{n-1}{n} + n^2 \cdot \frac{1}{n} \\ &= n + 1 - \frac{1}{n} \\ &\geq n. \end{aligned}$$

- Prove that
$$\mathbb{E}\left(X^2\right) \neq O\left((\mathbb{E}(X))^2\right).$$

Solution: We have seen above that

$$\mathbb{E}(Z) = \mathbb{E}\left(X^2\right) \geq n$$

and $\mathbb{E}(X) \leq 2$, implying that

$$(\mathbb{E}(X))^2 \leq 4.$$

It follows that

$$\frac{\mathbb{E}\left(X^2\right)}{(\mathbb{E}(X))^2} \geq \frac{n}{4} \rightarrow \infty$$

if $n \rightarrow \infty$. This is equivalent to what we were asked to prove.

Remark: A common mistake is to assume that $\mathbb{E}\left(X^2\right)$ is equal to $(\mathbb{E}(X))^2$. This example shows that $\mathbb{E}\left(X^2\right)$ can be much larger than $(\mathbb{E}(X))^2$.

- By the arguments above, Elisa gets, on average, a very large amount of cider. Since she cannot drink all these bottles, Carleton University changes their policy:

  1. The student chooses a uniformly random coin from the box (and puts it back in the box).
  2. The student chooses a uniformly random coin from the box (and puts it back in the box).
  3. If x is the number written on the first chosen coin, and y is the number written on the second chosen coin, then the student gives $\min(x^2, y^2)$ bottles of cider to Elisa.

  Consider the random variables

$$\begin{aligned} U &= \text{the number written on the first chosen coin,} \\ V &= \text{the number written on the second chosen coin,} \\ W &= \text{the number of bottles of cider that Elisa gets.} \end{aligned}$$

Use Question 5 to prove that

$$\mathbb{E}(W) \leq 4.$$

Solution: From the first part, we see that

$$\mathbb{E}(U) = \mathbb{E}(V) \leq 2.$$

Since

$$W = \min\left(U^2, V^2\right),$$

we can apply the result in Question 5, and get

$$\begin{aligned} \mathbb{E}(W) &= \mathbb{E}\left(\min\left(U^2, V^2\right)\right) \\ &\leq (\mathbb{E}(U))^2 \\ &\leq 2^2 \\ &= 4. \end{aligned}$$

Remark: The "trick" above has been used in the design of algorithms in computational geometry:

- If we make one choice (i.e., choose only one coin), then the "running time" is very high (Elisa gets, on average, at least n bottles).
- If we make two choices (i.e., choose two coins but keep the smaller one), then the "running time" is low (Elisa gets, on average, at most 4 bottles).

Question 7: Let $m \geq 1$, $n \geq 1$, and $k \geq 1$ be integers with $k \leq m + n$. Consider a set P consisting of m men and n women. We choose a uniformly random k-element subset Q of P. Consider the random variables

$$\begin{aligned} X &= \text{the number of men in the chosen subset } Q, \\ Y &= \text{the number of women in the chosen subset } Q, \\ Z &= X - Y. \end{aligned}$$

- Prove that
$$\mathbb{E}(Z) = 2 \cdot \mathbb{E}(X) - k.$$

  Solution: Since the subset Q has size k, we have $X + Y = k$ and, thus,
$$Z = X - Y = X - (k - X) = 2X - k,$$
  implying that
$$\begin{aligned} \mathbb{E}(Z) &= \mathbb{E}(2X - k) \\ &= 2 \cdot \mathbb{E}(X) - \mathbb{E}(k) \\ &= 2 \cdot \mathbb{E}(X) - k. \end{aligned}$$

- Determine the expected value $\mathbb{E}(X)$.

  Hint: Denote the men as $M_1, M_2, \ldots, M_m$. Use indicator random variables.

  Solution: For $i = 1, 2, \ldots, m$, define the indicator random variable
$$X_i = \begin{cases} 1 & \text{if } M_i \in Q, \\ 0 & \text{otherwise.} \end{cases}$$
  Then
$$X = \sum_{i=1}^{m} X_i$$

and

$$\begin{aligned}
\mathbb{E}(X) &= \mathbb{E}\left(\sum_{i=1}^{m} X_i\right) \\
&= \sum_{i=1}^{m} \mathbb{E}\left(X_i\right) \\
&= \sum_{i=1}^{m} \Pr\left(X_i = 1\right) \\
&= \sum_{i=1}^{m} \Pr\left(M_i \in Q\right).
\end{aligned}$$

To determine $\Pr\left(M_i \in Q\right)$, we observe that

1. the number of possible subsets Q is equal to $\binom{m+n}{k}$,
2. the number of subsets Q that contain M_i is equal to $\binom{m+n-1}{k-1}$.

It follows that

$$\begin{aligned}
\Pr\left(M_i \in Q\right) &= \frac{\binom{m+n-1}{k-1}}{\binom{m+n}{k}} \\
&= \frac{(m+n-1)!}{(k-1)!(m+n-k)!} \cdot \frac{k!(m+n-k)!}{(m+n)!} \\
&= \frac{k}{m+n}.
\end{aligned}$$

We conclude that

$$\mathbb{E}(X) = \sum_{i=1}^{m} \frac{k}{m+n} = \frac{km}{m+n}.$$

- Prove that

$$\mathbb{E}(Z) = k \cdot \frac{m-n}{m+n}.$$

Solution: If we put everything together that we have seen above, then we get

$$\begin{aligned}
\mathbb{E}(Z) &= 2 \cdot \mathbb{E}(X) - k \\
&= 2 \cdot \frac{km}{m+n} - k \\
&= \frac{2km - k(m+n)}{m+n} \\
&= k \cdot \frac{m-n}{m+n}.
\end{aligned}$$

Question 8: For any integer $n \geq 0$ and any real number x with $0 < x < 1$, define the function

$$F_n(x) = \sum_{k=n}^{\infty} \binom{k}{n} x^k.$$

(Using the *ratio test* from calculus, it can be shown that this infinite series converges for any fixed integer n.)

- Determine a closed form expression for $F_0(x)$. (You may use any result that was proven in class.)

- Let $n \geq 1$ be an integer and let x be a real number with $0 < x < 1$. Prove that

  $$F_n(x) = \frac{x}{n} \cdot F_{n-1}(x) + \frac{x^2}{n} \cdot F'_{n-1}(x),$$

  where F'_{n-1} denotes the derivative of F_{n-1}.

  Hint: If $k \geq n \geq 1$, then $\binom{k}{n} = \frac{k}{n}\binom{k-1}{n-1}$.

- Prove that for any integer $n \geq 0$ and any real number x with $0 < x < 1$,

  $$F_n(x) = \frac{x^n}{(1-x)^{n+1}},$$

  and

  $$F'_n(x) = \frac{x^n + n \cdot x^{n-1}}{(1-x)^{n+2}}.$$

- Let $n \geq 0$ and m be integers with $m \geq n+1$. Prove that

  $$\sum_{\ell=0}^{\min(n+1,m-n)} (-1)^{\ell} \binom{n+1}{\ell} \binom{m-\ell}{n} = 0.$$

  Hint: You have shown above that

  $$(1-x)^{n+1} \sum_{k=n}^{\infty} \binom{k}{n} x^k = (1-x)^{n+1} \cdot F_n(x) = x^n. \tag{1}$$

  Use Newton's Binomial Theorem to expand $(1-x)^{n+1}$. Then consider the expansion of the left-hand side in (1). What is the coefficient of x^m in this expansion?

Solution:
First part: The definition of $F_n(x)$, with $n = 0$, implies that

$$F_0(x) = \sum_{k=0}^{\infty} \binom{k}{0} x^k = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x};$$

the last step we have seen in class.
Second part: We start by using the hint:

$$\begin{aligned} F_n(x) &= \sum_{k=n}^{\infty} \binom{k}{n} x^k \\ &= \sum_{k=n}^{\infty} \frac{k}{n} \cdot \binom{k-1}{n-1} x^k. \end{aligned}$$

We do a change of summation variable: $\ell = k - 1$. Thus, we replace all occurrences of k by $1 + \ell$. Since $k = n, n + 1, n + 2, \ldots$, we have $\ell = n - 1, n, n + 1, \ldots$. This gives

$$\begin{aligned} F_n(x) &= \sum_{\ell=n-1}^{\infty} \frac{1+\ell}{n} \cdot \binom{\ell}{n-1} x^{1+\ell} \\ &= \frac{x}{n} \sum_{\ell=n-1}^{\infty} (1+\ell) \cdot \binom{\ell}{n-1} x^{\ell} \\ &= \frac{x}{n} \sum_{\ell=n-1}^{\infty} \binom{\ell}{n-1} x^{\ell} + \frac{x}{n} \sum_{\ell=n-1}^{\infty} \ell \cdot \binom{\ell}{n-1} x^{\ell}. \end{aligned}$$

The first summation on the right-hand side is equal to $F_{n-1}(x)$. If you take the summation for $F_{n-1}(x)$, differentiate it, and multiply by x, then you get the second summation on the right-hand size. We conclude that

$$\begin{aligned} F_n(x) &= \frac{x}{n} \cdot F_{n-1}(x) + \frac{x}{n} \cdot x \cdot F'_{n-1}(x) \\ &= \frac{x}{n} \cdot F_{n-1}(x) + \frac{x^2}{n} \cdot F'_{n-1}(x). \end{aligned}$$

Third part: We have to show that for all integers $n \geq 0$ and all real numbers x with $0 < x < 1$,

$$F_n(x) = \frac{x^n}{(1-x)^{n+1}},$$

and

$$F'_n(x) = \frac{x^n + n \cdot x^{n-1}}{(1-x)^{n+2}}.$$

The proof will be by induction on n.
Base case: We have seen above that

$$F_0(x) = \frac{1}{1-x} = \frac{x^0}{(1-x)^{0+1}}.$$

By taking the derivative of $\frac{1}{1-x}$, we get

$$F'_0(x) = \frac{1}{(1-x)^2} = \frac{x^0 + 0 \cdot x^{0-1}}{(1-x)^{0+2}}.$$

This proves the base case.
Induction step: Let $n \geq 1$ and assume that the claim is true for $n-1$. Thus, we assume that

$$F_{n-1}(x) = \frac{x^{n-1}}{(1-x)^n},$$

and

$$F'_{n-1}(x) = \frac{x^{n-1} + (n-1) \cdot x^{n-2}}{(1-x)^{n+1}}.$$

Using the recurrence for $F_n(x)$, the induction hypothesis, and some algebra, we get

$$\begin{aligned}
F_n(x) &= \frac{x}{n} \cdot F_{n-1}(x) + \frac{x^2}{n} \cdot F'_{n-1}(x) \\
&= \frac{x}{n} \cdot \frac{x^{n-1}}{(1-x)^n} + \frac{x^2}{n} \cdot \frac{x^{n-1} + (n-1) \cdot x^{n-2}}{(1-x)^{n+1}} \\
&= \frac{x^n}{n(1-x)^n} + \frac{x^{n+1} + (n-1)x^n}{n(1-x)^{n+1}} \\
&= \frac{x^n(1-x) + x^{n+1} + (n-1)x^n}{n(1-x)^{n+1}} \\
&= \frac{(x^n - x^{n+1}) + x^{n+1} + (n \cdot x^n - x^n)}{n(1-x)^{n+1}} \\
&= \frac{n \cdot x^n}{n(1-x)^{n+1}} \\
&= \frac{x^n}{(1-x)^{n+1}}.
\end{aligned}$$

I am sure you all remember from calculus that

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}.$$

In the above expression for $F_n(x)$, we take the derivative on both sides and do some algebra:

$$\begin{aligned}
F'_n(x) &= \text{derivative of } \frac{x^n}{(1-x)^{n+1}} \\
&= \frac{n \cdot x^{n-1}(1-x)^{n+1} + x^n \cdot (n+1)(1-x)^n}{(1-x)^{2n+2}} \\
&= \frac{n \cdot x^{n-1}(1-x) + x^n(n+1)}{(1-x)^{n+2}} \\
&= \frac{(n \cdot x^{n-1} - n \cdot x^n) + (n \cdot x^n + x^n)}{(1-x)^{n+2}} \\
&= \frac{x^n + n \cdot x^{n-1}}{(1-x)^{n+2}}.
\end{aligned}$$

Remark: In Question 9, we need the values of $F_n(1/4)$ and $F_n'(1/4)$. If you know an easier/shorter way to get these values, please let me know.
Fourth part: Let $n \geq 0$ and m be integers with $m \geq n+1$. We are going to show that

$$\sum_{\ell=0}^{\min(n+1,m-n)} (-1)^{\ell} \binom{n+1}{\ell} \binom{m-\ell}{n} = 0.$$

We have seen above that

$$F_n(x) = \sum_{k=n}^{\infty} \binom{k}{n} x^k = \frac{x^n}{(1-x)^{n+1}},$$

which is equivalent to

$$(1-x)^{n+1} \sum_{k=n}^{\infty} \binom{k}{n} x^k = x^n.$$

Sir Isaac Newton tells us that

$$(1-x)^{n+1} = \sum_{\ell=0}^{n+1} \binom{n+1}{\ell} (-x)^{\ell},$$

implying that

$$\left(\sum_{\ell=0}^{n+1} \binom{n+1}{\ell} (-x)^{\ell} \right) \cdot \left(\sum_{k=n}^{\infty} \binom{k}{n} x^k \right) = x^n.$$

Note that the coefficient of x^m on the left-hand side is equal to the coefficient of x^m on the right-hand side. Since $m \geq n+1$, the coefficient of x^m on the right-hand side is equal to 0.

How do we get a term x^m on the left-hand side:

1. The left-hand side is a product of two series.
2. In the first product, take the ℓ-th term.
3. In the second product, take the k-th term, where $k = m - \ell$.
4. If we multiply these terms, we get

$$\binom{n+1}{\ell} (-x)^{\ell} \cdot \binom{m-\ell}{n} x^{m-\ell} = (-1)^{\ell} \binom{n+1}{\ell} \binom{m-\ell}{n} x^m.$$

5. What are the possible values for ℓ:
   (a) The summation over ℓ starts at $\ell = 0$. Therefore, $\ell \geq 0$.
   (b) The summation over ℓ ends at $\ell = n+1$. Therefore, $\ell \leq n+1$.

(c) The summation over k starts at $k = n$. Therefore, $k \geq n$. Since $k = m - \ell$, we get $m - \ell \geq n$, which is equivalent to $\ell \leq m - n$.

(d) We conclude that
$$0 \leq \ell \leq \min(n+1, m-n).$$

This shows that the coefficient of x^m on the left-hand side is equal to

$$\sum_{\ell=0}^{\min(n+1,m-n)} (-1)^\ell \binom{n+1}{\ell} \binom{m-\ell}{n}.$$

Since this is equal to 0, we are done.

Question 9: Consider a fair red coin and a fair blue coin. We repeatedly flip both coins, and keep track of the number of times that the red coin comes up heads. As soon as the blue coin comes up tails, the process terminates.

A formal description of this process is given in the pseudocode below. The value of the variable i is equal to the number of iterations performed so far, the value of the variable h is equal to the number of times that the red coin came up heads so far, whereas the Boolean variable *stop* is used to decide when the while-loop terminates.

```
Algorithm RandomCoinFlips:

    // both the red coin and the blue coin are fair
    // all coin flips are mutually independent
    i = 0;
    h = 0;
    stop = false;
    while stop = false
    do i = i + 1;
       flip the red coin;
       if the result of the red coin is heads
       then h = h + 1
       endif;
       flip the blue coin;
       if the result of the blue coin is tails
       then stop = true
       endif
    endwhile;
    return i and h
```

Consider the random variables

X = the value of i that is returned by algorithm RandomCoinFlips,

Y = the value of h that is returned by algorithm RandomCoinFlips.

Assume that the value of the random variable Y is equal to some integer n. In this exercise, you will determine the expected value of the random variable X.

Thus, we are interested in the *conditional expected value* $\mathbb{E}(X \mid Y = n)$, which is the expected value of X (i.e., the number of iterations of the while-loop), when you are given that the event "$Y = n$" (i.e., during the while-loop, the red coin comes up heads n times) occurs. Formally, we have

$$\mathbb{E}(X \mid Y = n) = \sum_k k \cdot \Pr(X = k \mid Y = n),$$

where the summation ranges over all values of k that X can take.

For the rest of this exercise, let $n \geq 1$ be an integer. The functions F_n and F'_n that are used below are the same as those in Question 8.

- Prove that
$$\Pr(Y = n) = \sum_{k=n}^{\infty} \Pr(Y = n \mid X = k) \cdot \Pr(X = k).$$
- Prove that
$$\Pr(Y = n) = F_n(1/4).$$
- Prove that
$$\mathbb{E}(X \mid Y = n) = \frac{F'_n(1/4)}{4 \cdot F_n(1/4)}.$$
- Prove that
$$\mathbb{E}(X \mid Y = n) = \frac{4n + 1}{3}.$$

Solution:
First part: The event "$Y = n$" states that the number of red heads is equal to n. For this to happen, the number of iterations must be at least equal to n. (Note that the number of iterations is always at least 1. Because of this, we need that $n \geq 1$ in the rest of these solutions.) If we denote the number of iterations by k, then

$$Y = n \Longleftrightarrow \bigvee_{k=n}^{\infty} (Y = n \wedge X = k),$$

implying that

$$\Pr(Y = n) = \Pr\left(\bigvee_{k=n}^{\infty} (Y = n \wedge X = k)\right).$$

Since the events on the right-hand side are pairwise disjoint, we get

$$\Pr(Y = n) = \sum_{k=n}^{\infty} \Pr(Y = n \wedge X = k).$$

Using the definition of conditional probability, we get

$$\Pr(Y = n) = \sum_{k=n}^{\infty} \Pr(Y = n \mid X = k) \cdot \Pr(X = k).$$

Second part: To determine $\Pr(Y = n)$, we determine the terms on the right-hand side in the above summation:

1. The event "$X = k$" happens if and only if the first $k-1$ blue coin flips result in heads, and the k-th blue flip results in tails. This implies that

$$\Pr(X = k) = \Pr\left(H^{k-1}T\right) = (1/2)^k.$$

2. To determine $\Pr(Y = n \mid X = k)$, we assume that the event "$X = k$" happens. This means that the number of iterations is equal to k, i.e., we flip the red coin k times. The event "$Y = n$" says that the number of red heads is equal to n. Since there are $\binom{k}{n}$ ways in which this can happen, we get

$$\Pr(Y = n \mid X = k) = \binom{k}{n}(1/2)^k.$$

From this, we get

$$\begin{aligned}
\Pr(Y = n) &= \sum_{k=n}^{\infty} \Pr(Y = n \mid X = k) \cdot \Pr(X = k) \\
&= \sum_{k=n}^{\infty} \binom{k}{n}(1/2)^k \cdot (1/2)^k \\
&= \sum_{k=n}^{\infty} \binom{k}{n}(1/4)^k \\
&= F_n(1/4).
\end{aligned}$$

Third part: We take the definition of $\mathbb{E}(X \mid Y = n)$, use the definition of conditional probability to rewrite the summation, and plug in the values that we have seen so far:

$$\begin{aligned}
\mathbb{E}(X \mid Y = n) &= \sum_{k=n}^{\infty} k \cdot \Pr(X = k \mid Y = n) \\
&= \sum_{k=n}^{\infty} k \cdot \frac{\Pr(X = k \wedge Y = n)}{\Pr(Y = n)} \\
&= \frac{1}{\Pr(Y = n)} \sum_{k=n}^{\infty} k \cdot \Pr(Y = n \mid X = k) \cdot \Pr(X = k)
\end{aligned}$$

$$= \frac{1}{F_n(1/4)} \sum_{k=n}^{\infty} k \cdot \binom{k}{n} (1/2)^k \cdot (1/2)^k$$

$$= \frac{1}{F_n(1/4)} \sum_{k=n}^{\infty} k \cdot \binom{k}{n} (1/4)^k.$$

How do we simplify the summation on the right-hand side: If we take the definition of $F_n(x)$, differentiate it, and multiply by x, we get

$$x \cdot F_n'(x) = \sum_{k=n}^{\infty} k \binom{k}{n} x^k.$$

It follows that the summation on the right-hand side is equal to

$$1/4 \cdot F_n'(1/4).$$

We conclude that

$$\mathbb{E}(X \mid Y = n) = \frac{1}{F_n(1/4)} \cdot 1/4 \cdot F_n'(1/4) = \frac{F_n'(1/4)}{4 \cdot F_n(1/4)}.$$

Fourth part: We next show that

$$\mathbb{E}(X \mid Y = n) = \frac{4n+1}{3}.$$

From Question 8, we get

$$\begin{aligned} \frac{F_n'(x)}{F_n(x)} &= \frac{\frac{x^n + n \cdot x^{n-1}}{(1-x)^{n+2}}}{\frac{x^n}{(1-x)^{n+1}}} \\ &= \frac{x^n + n \cdot x^{n-1}}{x^n(1-x)} \\ &= \frac{x+n}{x(1-x)}. \end{aligned}$$

We conclude that

$$\begin{aligned} \mathbb{E}(X \mid Y = n) &= \frac{1}{4} \cdot \frac{F_n'(1/4)}{F_n(1/4)} \\ &= \frac{1}{4} \cdot \frac{1/4 + n}{1/4 \cdot 3/4} \\ &= \frac{4n+1}{3}. \end{aligned}$$