# Midterm COMP 2804

October 23, 2013

- All questions must be answered on the scantron sheet.
- Write your name and student number on the scantron sheet.
- You do not have to hand in this examination paper.

**Marking scheme:** Each of the 17 questions is worth 1 mark.

- Newton: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$.

1. Let $A$ be a set of size 7 and let $B$ be a set of size 13. How many one-to-one functions $f : A \to B$ are there?

   (a) $\frac{6!}{13!}$
   (b) $\frac{13!}{5!}$
   (c) $\frac{13!}{6!}$
   (d) $\frac{13!}{7!}$

2. You are given 5 books and 7 bookshelves. How many ways are there to place these books on the shelves? (The order on the shelves matters.)

   (a) $\binom{7}{5}$
   (b) $\frac{11!}{6!}$
   (c) $\frac{11!}{7!}$
   (d) $\frac{12!}{7!}$

3. A password consists of 6 or 7 characters, each character being an uppercase letter or a lowercase letter. A password must contain at least one uppercase letter. How many passwords are there?

   (a) $52^6 + 52^7$
   (b) $26 \cdot 52^5 + 26 \cdot 52^6$
   (c) $52^6 + 52^7 - 26^6 - 26^7$
   (d) none of the above

4. In a group of 20 people,

   - 6 are blond,
   - 7 have green eyes,
   - 11 are not blond and do not have green eyes.

   How many people are blond and have green eyes?

   (a) 3
   (b) 4
   (c) 5
   (d) 9

5. How many bitstrings of length 55 start with 101 or end with 1111?

   (a) $2^{52} + 2^{51}$

   (b) $2^{55} - 2^{48}$

   (c) $2^{55} - 2^{52} - 2^{51}$

   (d) $2^{52} + 2^{51} - 2^{48}$

6. Each person in a group of $n$ people has a last name consisting of two uppercase letters. For what values of $n$ can we guarantee that there are at least two people with the same last name?

   (a) $n \geq 26$

   (b) $n \geq 52$

   (c) $n \geq 676$

   (d) $n \geq 677$

7. How many bitstrings of length 13 contain exactly 3 zeros?

   (a) $\binom{13}{10}$

   (b) $13!/3!$

   (c) $2^{13} - \binom{13}{3}$

   (d) $2^{13} - 3$

8. What is the coefficient of $x^{12}y^{12}$ in the expansion of $(3x - 7y)^{24}$?

   (a) $-3^{12}7^{12}\binom{24}{12}$

   (b) $(3x)^{12}(-7y)^{12}\binom{24}{12}$

   (c) $21^{12}\binom{24}{12}$

   (d) $(3x)^{12}(7y)^{12}\binom{24}{12}$

9. Which of the following is true?

   (a) $\sum_{k=0}^{n} 5^k \binom{n}{k} = 6^n$

   (b) $\sum_{k=0}^{n} 4^{n-k}5^k \binom{n}{k} = 8^n$

   (c) $\sum_{k=0}^{n} 5^k \binom{n}{k} = 5^n$

   (d) $\sum_{k=0}^{n} 4^k 5^{n-k} \binom{n}{k} = 20^n$

10. How many strings can be obtained by rearranging the letters of the word

POOPERSCOOPER

(a) 13!

(b) $\binom{13}{4}\binom{9}{3}\binom{6}{2}\binom{4}{2}\binom{2}{1}$

(c) $\binom{13}{4}\binom{9}{3}\binom{6}{2}\binom{4}{2}$

(d) 4!3!2!2!1!1!

11. The function $f : \mathbb{N} \to \mathbb{N}$ is defined by

$$\begin{aligned} f(0) &= 14 \\ f(n+1) &= f(n) + 4n - 5 \text{ for } n \geq 0 \end{aligned}$$

What is $f(n)$?

(a) $f(n) = 2n^2 + 6n + 14$

(b) $f(n) = 2n^2 - 6n + 14$

(c) $f(n) = 2n^2 + 7n + 14$

(d) $f(n) = 2n^2 - 7n + 14$

12. Consider the following recursive algorithm FIB, which takes as input an integer $n \geq 0$:

**Algorithm** FIB($n$):
**if** $n = 0$ or $n = 1$
**then** $f = n$
**else** $f =$ FIB$(n-1) +$ FIB$(n-2)$
**endif**;
return $f$

When running FIB(5), how many calls are there to FIB(2)?

(a) 1

(b) 2

(c) 3

(d) 4

13. The Fibonacci numbers are defined as follows: $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$.
Consider again the recursive algorithm FIB, which takes as input an integer $n \geq 0$:

    **Algorithm** FIB($n$):
    **if** $n = 0$ or $n = 1$
    **then** $f = n$
    **else** $f =$ FIB$(n-1) +$ FIB$(n-2)$
    **endif**;
    return $f$

    For $n \geq 2$, run algorithm FIB($n$) and let $a_n$ be the number of times that FIB(0) is called.

    (a) For $n \geq 2$, $a_n = f_{n-1}$
    (b) For $n \geq 2$, $a_n = f_n$
    (c) For $n \geq 2$, $a_n = f_n - 1$
    (d) For $n \geq 2$, $a_n = f_{n+1}$

14. What does the summation $\sum_{k=7}^{n} \binom{k-1}{6}$ count?

    (a) The number of subsets of $\{1, 2, \ldots, n\}$ having size 5.
    (b) The number of subsets of $\{1, 2, \ldots, n\}$ having size 6.
    (c) The number of subsets of $\{1, 2, \ldots, n\}$ having size 7.
    (d) The number of pints of beer you drink when running algorithm BEER($n$).

15. If you flip a fair coin 4 times, what is the probability that the coin comes up head exactly twice?

    (a) $1/\binom{4}{2}$
    (b) $2/2^4$
    (c) $2^4/\binom{4}{2}$
    (d) $\binom{4}{2}/2^4$

16. If you choose an element $x$ uniformly at random from the set $\{1, 2, \ldots, 100\}$, what is the probability that $x$ is divisible by 4 or 5?

    (a) 9/100
    (b) 1/5
    (c) 2/5
    (d) 45/100

17. If you answer each question in this midterm by choosing an answer uniformly at random, what is the probability that you get all answers correct?

    (a) $1/17^4$

    (b) $1/4^{17}$

    (c) $3^{17}/4^{17}$

    (d) $4^{17}/3^{17}$