Midterm COMP 2804

October 23, 2015

- All questions must be answered on the scantron sheet.
- Write your name and student number on the scantron sheet.
- You do not have to hand in this examination paper.
- Calculators are allowed.

Marking scheme: Each of the 17 questions is worth 1 mark.

- $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
- Newton: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$.

1. The Carleton Computer Science Society has a Board of Directors consisting of a President, two Vice-Presidents, and a five-person Advisory Board. The President cannot be Vice-President and cannot be on the Advisory Board. A Vice-President cannot be on the Advisory Board. Let n be the number of students in Carleton's Computer Science program, where $n \geq 8$. In how many ways can a Board of Directors be chosen?

   (a) $n\binom{n}{2}\binom{n}{5}$
   (b) $(n-2)\binom{n}{2}\binom{n-2}{5}$
   (c) $(n-5)\binom{n}{2}\binom{n-1}{5}$
   (d) $(n-7)\binom{n}{2}\binom{n-2}{5}$

2. Let S be a set of 25 elements and let x, y, and z be three distinct elements of S. What is the number of subsets of S that contain both x and y, but do not contain z?

   (a) $2^{25} - 2^{22}$
   (b) $2^{25} - 2^{24} + 2^{23}$
   (c) 2^{22}
   (d) 2^{23}

3. Let A be a set of 6 elements and let B be a set of 13 elements. How many one-to-one (i.e., injective) functions $f : A \to B$ are there?

   (a) $5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13$
   (b) $6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13$
   (c) $7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13$
   (d) $8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13$

4. For any integer $n \geq 2$, let S_n be the number of bitstrings of length n in which the first bit is not equal to the last bit. Which of the following is true?

   (a) $S_n = 2^{n-2}$
   (b) $S_n = 2^{n-1}$
   (c) $S_n = 2^n - 2^{n-2}$
   (d) $S_n = 2^n - 2^{n-1} + 2^{n-2}$

5. Consider strings of length 99 consisting of the characters a, b, and c. How many such strings are there that start with abc or end with bbb?

   (a) $3^{96} + 3^{96}$

   (b) $3^{99} - 2 \cdot 3^{96}$

   (c) $2 \cdot 3^{96} - 3^{93}$

   (d) None of the above.

6. What does

$$\sum_{k=1}^{m} \binom{m}{k}$$

   count?

   (a) The number of non-empty subsets of a set of size m.

   (b) The number of subsets of a set of size m.

   (c) The number of bitstrings of length m having exactly k many 1s.

   (d) None of the above.

7. In the city of ShortLastName, every person has a last name consisting of two characters, the first one being an uppercase letter and the second one being a lowercase letter. What is the minimum number of people needed so that we can guarantee that at least 4 of them have the same last name?

   (a) $3 \cdot 26^2$

   (b) $4 \cdot 26^2$

   (c) $3 \cdot 26^2 + 1$

   (d) $4 \cdot 26^2 + 1$

8. What is the coefficient of $x^{81}y^7$ in the expansion of $(3x - 17y)^{88}$?

   (a) $\binom{88}{7} \cdot 3^{81} \cdot 17^7$

   (b) $-\binom{88}{7} \cdot 3^{81} \cdot 17^7$

   (c) $\binom{88}{7} \cdot 3^7 \cdot 17^{81}$

   (d) $-\binom{88}{7} \cdot 3^7 \cdot 17^{81}$

9. How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 55$, where $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$, and $x_4 \geq 0$ are integers?

(a) $\binom{58}{3}$

(b) $\binom{58}{4}$

(c) $\binom{59}{3}$

(d) $\binom{59}{4}$

10. The function $f : \mathbb{N} \to \mathbb{N}$ is defined by

$$\begin{aligned} f(0) &= 7 \\ f(n) &= f(n-1) + 10n - 6 \text{ for } n \geq 1 \end{aligned}$$

What is $f(n)$?

(a) $f(n) = 4n^2 - 2n + 7$

(b) $f(n) = 4n^2 - n + 7$

(c) $f(n) = 5n^2 - 2n + 7$

(d) $f(n) = 5n^2 - n + 7$

11. Let S_n be the number of bitstrings of length n that contain the substring 0000. Which of the following is true?

(a) $S_n = S_{n-1} + S_{n-2} + S_{n-3} + S_{n-4}$

(b) $S_n = S_{n-1} + S_{n-2} + S_{n-3} + S_{n-4} + 2^{n-4}$

(c) $S_n = S_{n-1} + S_{n-2} + S_{n-3}$

(d) $S_n = S_{n-1} + S_{n-2} + S_{n-3} + 2^{n-3}$

12. Let $n \geq 1$ be an integer and let S_n be the number of ways in which n can be written as a sum of 1s and 2s, such that

- the order in which the 1s and 2s occur in the sum matters, and
- it is not allowed to have two consecutive 2s.

For example, if $n = 7$, then both

$$7 = 1 + 2 + 1 + 2 + 1$$

and

$$7 = 2 + 1 + 1 + 2 + 1$$

are allowed, whereas

$$7 = 1 + 2 + 2 + 1 + 1$$

is not allowed.
Which of the following is true?

(a) $S_n = S_{n-1} + S_{n-2}$
(b) $S_n = S_{n-1} + S_{n-3}$
(c) $S_n = S_{n-2} + S_{n-3}$
(d) $S_n = S_{n-1} + S_{n-2} + S_{n-3}$

13. Consider the following recursive algorithm FIB, which takes as input an integer $n \geq 0$:

Algorithm FIB(n):
if $n = 0$ or $n = 1$
then $f = n$
else $f = $ FIB$(n-1)$ + FIB$(n-2)$
endif;
return f

When running FIB(55), how many calls are there to FIB(50)?

(a) 6
(b) 7
(c) 8
(d) 9

14. Consider the following recursive algorithm JUSTINBIEBER, which takes as input an integer $n \geq 1$, which is a power of 2:

```
Algorithm JustinBieber(n):
if n = 1
then order chicken wings
else if n = 2
     then drink one pint of India Pale Ale
     else print "I don't like Justin Bieber";
          JustinBieber(n/2)
     endif
endif
```

For n a power of 2, let $B(n)$ be the number of times you print "I don't like Justin Bieber" when running algorithm JUSTINBIEBER(n). Which of the following is true?

(a) $B(n) = \log n - 1$ for all $n \geq 2$.

(b) $B(n) = \log n - 1$ for all $n \geq 1$.

(c) $B(n) = \log n$ for all $n \geq 2$.

(d) $B(n) = n - 2$ for all $n \geq 2$.

15. You flip a fair coin 7 times. Define the event

$$A = \text{"the result of the first flip is equal to the result of the 7-th flip".}$$

What is $\Pr(A)$?

(a) $\frac{2^5+2}{2^7}$

(b) 1/2

(c) 1/3

(d) 1/4

16. You roll a fair 6-sided die twice. Define the events

$$A = \text{“the sum of the results of the two rolls is 7”}$$

and

$$B = \text{“the result of the first roll is 3”}.$$

Which of the following is true?

(a) $\Pr(A) = \Pr(B)$
(b) $\Pr(A) < \Pr(B)$
(c) $\Pr(A) > \Pr(B)$
(d) None of the above.

17. Let $S = \{1, 2, 3, 4, 5, 6, 7\}$. You choose a uniformly random 3-element subset X of S. Thus, each 3-element subset of S has a probability of $1/\binom{7}{3}$ of being X. Define the event

$$A = \text{“4 is an element of } X\text{”}$$

What is $\Pr(A)$?

(a) 7/15
(b) 15/7
(c) 3/7
(d) 7/3