Midterm COMP 2804

November 2, 2016

- All questions must be answered on the scantron sheet.
- Write your name and student number on the scantron sheet.
- You do not have to hand in this examination paper.
- Calculators are allowed.

Marking scheme: Each of the 17 questions is worth 1 mark.

- $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
- Newton: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$.

1. Carleton's Computer Science program has f female students and m male students, where $f \geq 1$ and $f+m \geq 4$. The Carleton Computer Science Society has a Board of Directors consisting of a President and three Vice-Presidents, all of whom are Computer Science students. The President is female and cannot be Vice-President. In how many ways can a Board of Directors be chosen?

   (a) $\binom{f+m}{4}$
   (b) $f \cdot \binom{f+m-1}{3}$
   (c) $f \cdot \binom{f+m}{3}$
   (d) $(f-1) \cdot \binom{f+m}{3}$

2. Let k and n be integers with $2 \leq k \leq n$ and consider the set $S = \{1, 2, \ldots, n\}$. What is the number of k-element subsets of S that do not contain 1 *and* do not contain 2?

   (a) $\binom{n-1}{k-1}$
   (b) $\binom{n-1}{k}$
   (c) $\binom{n-2}{k-2}$
   (d) $\binom{n-2}{k}$

3. Let k and n be integers with $2 \leq k \leq n$ and consider the set $S = \{1, 2, \ldots, n\}$. What is the number of k-element subsets of S that do not contain 1 *or* do not contain 2?

   (a) $\binom{n-1}{k} + \binom{n-1}{k}$
   (b) $\binom{n-2}{k}$
   (c) $\binom{n}{k} - \binom{n-2}{k-2}$
   (d) $\binom{n}{k} - \binom{n-1}{k-1} - \binom{n-1}{k-1}$

4. For any integer $n \geq 3$, let B_n be the number of bitstrings of length n in which the first three bits are not all equal. Which of the following is true?

   (a) $B_n = 2 \cdot 2^{n-3}$
   (b) $B_n = 6 \cdot 2^{n-3}$
   (c) $B_n = 2^n - 2$
   (d) $B_n = 2^n - 6$

5. Consider strings of length 4 consisting of the characters a, b, c, and d. How many such strings are there that start with ad or end with dcb?

   (a) 17
   (b) 18
   (c) 19
   (d) 20

6. Let $n \geq 5$ and consider strings of length n consisting of the characters a, b, c, and d. How many such strings are there that start with ad or end with dcb?

   (a) $4^{n-2} + 4^{n-3} - 4^{n-5}$
   (b) $4^{n-2} + 4^{n-3}$
   (c) $4^n - 4^{n-5}$
   (d) $4^n - 4^{n-2} - 4^{n-3}$

7. What does

$$\binom{w}{3} \cdot \binom{m}{2} + \binom{w}{4} \cdot m + \binom{w}{5}$$

   count?

   (a) The number of ways to choose 5 people out of a group consisting of w women and m men, where at most 3 women can be chosen.
   (b) The number of ways to choose 5 people out of a group consisting of w women and m men, where at most 3 men can be chosen.
   (c) The number of ways to choose 5 people out of a group consisting of w women and m men, where at least 3 women must be chosen.
   (d) The number of ways to choose 5 people out of a group consisting of w women and m men, where at least 3 men must be chosen.

8. Let $n \geq 2$ be an integer and let S be a set of m integers. What is the minimum value of m such that we can guarantee that S contains at least two elements whose difference is divisible by n?

   (a) $n - 1$
   (b) n
   (c) $n + 1$
   (d) $n + 2$

9. What is the coefficient of $x^{24}y^{26}$ in the expansion of $(5x - 7y)^{50}$?

   (a) $-\binom{50}{24} \cdot 5^{26} \cdot 7^{24}$
   (b) $-\binom{50}{26} \cdot 5^{24} \cdot 7^{26}$
   (c) $\binom{50}{24} \cdot 5^{26} \cdot 7^{24}$
   (d) $\binom{50}{26} \cdot 5^{24} \cdot 7^{26}$

10. The function $f : \mathbb{N} \to \mathbb{R}$ is defined by

$$\begin{aligned} f(0) &= 7, \\ f(n) &= \frac{n}{3} \cdot f(n-1) \text{ for } n \geq 1. \end{aligned}$$

   What is $f(n)$?

   (a) $f(n) = 7 \cdot \frac{n!}{3^n}$
   (b) $f(n) = 7^n \cdot \frac{n!}{3^n}$
   (c) $f(n) = 7 \cdot \frac{(n+1)!}{3^n}$
   (d) $f(n) = 7^n \cdot \frac{(n+1)!}{3^n}$

11. For any integer $n \geq 1$, let B_n be the number of bitstrings of length n that do not contain the substring 11 and do not contain the substring 101. Which of the following is true for any $n \geq 4$?

   (a) $B_n = B_{n-1} + B_{n-2}$
   (b) $B_n = B_{n-1} + B_{n-3}$
   (c) $B_n = B_{n-2} + B_{n-3}$
   (d) $B_n = B_{n-2} + B_{n-4}$

12. Let $n \geq 1$ be an integer, and let S_n be the number of ways in which n can be written as a sum of 3s and 4s, such that the order in which the 3s and 4s occur in the sum matters. For example, $S_5 = 0$, because 5 cannot be written as a sum of 3s and 4s. We have $S_{11} = 3$, because

$$11 = 3 + 4 + 4 = 4 + 3 + 4 = 4 + 4 + 3.$$

   Which of the following is true for any $n \geq 5$?

   (a) $S_n = 2 \cdot S_{n-1}$
   (b) $S_n = S_{n-1} + S_{n-2}$
   (c) $S_n = S_{n-2} + S_{n-3}$
   (d) $S_n = S_{n-3} + S_{n-4}$

13. Consider the following recursive algorithm FIB, which takes as input an integer $n \geq 0$:

Algorithm FIB(n):
if $n = 0$ or $n = 1$
then $f = n$
else $f = \text{FIB}(n-1) + \text{FIB}(n-2)$
endif;
return f

When running FIB(12), how many calls are there to FIB(8)?

(a) 4
(b) 5
(c) 6
(d) 7

14. Consider the following recursive algorithm ELISADRINKSCIDER, which takes as input an integer $n \geq 1$, which is a power of 2:

Algorithm ELISADRINKSCIDER(n):
if $n = 1$
then order Fibonachos
else ELISADRINKSCIDER($n/2$);
drink n bottles of cider;
ELISADRINKSCIDER($n/2$)
endif

For n a power of 2, let $C(n)$ be the total number of bottles of cider that you drink when running algorithm ELISADRINKSCIDER(n). Which of the following is true for any $n \geq 1$ that is a power of 2?

(a) $C(n) = n \log n - 1$
(b) $C(n) = n \log n + 1$
(c) $C(n) = n \log n$
(d) $C(n) = 2n \log n$

15. You flip a fair coin 9 times. Define the event

$$A = \text{"the result of the first flip is not equal to the result of the second flip"}.$$

What is $\Pr(A)$?

(a) 1/4
(b) 1/3
(c) 1/2
(d) 1

16. Consider 4 people, each of which has a uniformly random birthday. We ignore leap years; thus, one year has 365 days. Define the event

$$A = \text{"at least 2 of these 4 people have the same birthday"}.$$

What is $\Pr(A)$?

(a) $\binom{4}{2} \cdot \frac{1}{365}$
(b) $\binom{4}{2} \cdot \frac{1}{365} + \binom{4}{3} \cdot \frac{1}{365^2} + \binom{4}{4} \cdot \frac{1}{365^3}$
(c) $1 - \frac{361 \cdot 362 \cdot 363 \cdot 364}{365^4}$
(d) $1 - \frac{362 \cdot 363 \cdot 364}{365^3}$

17. In the game of *Hearthstone*, you are given a deck of 18 distinct cards. One of the cards is the *Raven Idol*. You choose a uniformly random deck of 3 cards. Define the event

$$A = \text{"the hand of 3 cards contains the } \textit{Raven Idol}\text{"}.$$

What is $\Pr(A)$?

(a) 3/17
(b) 3/18
(c) 3/19
(d) 4/19