

Midterm COMP 2804

March 1, 2017

- All questions must be answered on the scantron sheet.
- Write your name and student number on the scantron sheet.
- You do not have to hand in this examination paper.
- Calculators are allowed.

Marking scheme: Each of the 17 questions is worth 1 mark.

- $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
- Newton: $(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$.
- The Fibonacci numbers are defined as follows: $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$.

1. Carleton's Computer Science program has f female students and m male students, where $f \geq 1$ and $f + m \geq 8$. The Carleton Computer Science Society has a Board of Directors consisting of a President, five Vice-Presidents, and two Cider-Presidents (whose task is to buy cider for the President). All members of the Board of Directors are Computer Science students. The President is female, cannot be Vice-President, and cannot be Cider-President. A Vice-President cannot be Cider-President. In how many ways can a Board of Directors be chosen?

- (a) $f \cdot \binom{f+m-1}{7}$
- (b) $f + \binom{f+m-1}{5} + \binom{f+m-6}{2}$
- (c) $f \cdot \binom{f+m-1}{5} \cdot \binom{f+m-6}{2}$
- (d) $f \cdot \binom{f+m}{5} \cdot \binom{f+m}{2}$

2. Let $b \geq 1$ and $g \geq 1$ be integers. Consider b boys and g girls. How many ways are there to arrange these kids on a line such that the leftmost kid is a girl?

- (a) $(b + g)!$
- (b) $g \cdot (b + g - 1)!$
- (c) $(b + g)!/b$
- (d) None of the above.

3. Let $b \geq 1$ and $g \geq 1$ be integers. Consider b boys and g girls. How many ways are there to arrange these kids on a line such that the leftmost kid is a girl or the rightmost kid is a boy?

- (a) $(b + g)!$
- (b) $(b + g)! - b \cdot (b + g - 1)! - g \cdot (b + g - 1)!$
- (c) $g \cdot (b + g - 1)! + b \cdot (b + g - 1)! - b \cdot g \cdot (b + g - 1)!$
- (d) $g \cdot (b + g - 1)! + b \cdot (b + g - 1)! - b \cdot g \cdot (b + g - 2)!$

4. Let $n \geq 13$ be an integer. What is the number of bitstrings of length n that have exactly seven 0's or exactly five 1's?

- (a) $\binom{n}{12} \binom{12}{7}$
- (b) $\binom{n}{7} + \binom{n}{5}$
- (c) $\binom{n}{7} + \binom{n}{5} - \binom{n}{5} \binom{n-5}{7}$
- (d) None of the above.

5. Consider strings of length six consisting of the characters a , b , c , and d . How many such strings are there that start with abc or end with $cccc$?
- (a) 79
 - (b) 80
 - (c) 81
 - (d) 82
6. Let $n \geq 4$ be an integer and consider strings of length n consisting of the characters a , b , c , and d . How many such strings are there that start with ab or start with abc ?
- (a) 4^n
 - (b) 4^{n-1}
 - (c) 4^{n-2}
 - (d) 4^{n-3}

7. Let $m \geq 2$ and $n \geq 2$ be integers. What does

$$\binom{m}{2} + \binom{n}{2} + mn$$

count?

- (a) The number of ways to choose an ordered pair (x, y) from a group consisting of m men and n women, where x must be a man and y must be a woman.
 - (b) The number of ways to choose an ordered pair (x, y) from a group consisting of m men and n women, where x and y cannot both be men.
 - (c) The number of ways to choose an unordered pair of people from a group consisting of m men and n women, where at least one man must be chosen.
 - (d) The number of ways to choose an unordered pair of people from a group consisting of m men and n women.
8. Each student has a student number which is a string of five digits, in which no digit occurs more than once. Thus, 94631 is a valid student number, whereas 94641 is not valid. What is the minimum number of students such that we can guarantee that at least two of them have the same student number?
- (a) $1 + 10^5$
 - (b) $1 + 5^{10}$
 - (c) $1 + \frac{10!}{5!}$
 - (d) $1 + \frac{5!}{10!}$

9. What is the coefficient of $x^{20}y^{80}$ in the expansion of $(7x - 13y)^{100}$?

- (a) $\binom{100}{80} \cdot 7^{20} \cdot 13^{80}$
- (b) $-\binom{100}{80} \cdot 7^{20} \cdot 13^{80}$
- (c) $\binom{100}{20} \cdot 7^{80} \cdot 13^{20}$
- (d) $-\binom{100}{20} \cdot 7^{80} \cdot 13^{20}$

10. How many strings can be obtained by rearranging the letters of the word

POOPERSCOOPER

- (a) $13!$
- (b) $\binom{13}{4} \binom{13}{3} \binom{13}{2} \binom{13}{1}$
- (c) $\binom{13}{4} \binom{9}{3} \binom{6}{2} \binom{4}{2}$
- (d) $\binom{13}{4} \binom{9}{3} \binom{6}{2} \binom{4}{2} \binom{2}{1}$

11. A bitstring is called *00-free* if it does not contain two 0's next to each other. In class, we have seen that for any $m \geq 1$, the number of 00-free bitstrings of length m is equal to the $(m + 2)$ -th Fibonacci number f_{m+2} .

Let $n \geq 3$ be an integer. What is the number of 00-free bitstrings of length $3n - 1$ that have 0 at position n and 1 at position $2n$? (The positions are numbered $1, 2, \dots, 3n - 1$.)

- (a) $f_{n+1}^2 \cdot f_n$
- (b) $f_n^2 \cdot f_{n+1}$
- (c) $f_{n+2}^2 \cdot f_{n+1}$
- (d) $f_{n+1}^2 \cdot f_{n+2}$

12. The function $f : \{1, 2, 3, \dots\} \rightarrow \mathbb{R}$ is defined by

$$\begin{aligned} f(1) &= 2, \\ f(n) &= 2 \left(\frac{n-1}{n} \right)^2 \cdot f(n-1) \text{ for } n \geq 2. \end{aligned}$$

What is $f(n)$?

- (a) $f(n) = \frac{2^n}{(n-1)^2}$
- (b) $f(n) = \frac{2^{n-1}}{n^2}$
- (c) $f(n) = \frac{2^n}{n^2}$
- (d) $f(n) = \frac{n^2}{2^n}$

13. For any integer $n \geq 1$, let B_n be the number of bitstrings of length n that do not contain the substring 10. Which of the following is true for any $n \geq 4$?

- (a) $B_n = n$
- (b) $B_n = n + 1$
- (c) $B_n = f_{n+1}$, the $(n + 1)$ -st Fibonacci number.
- (d) $B_n = f_{n+2}$, the $(n + 2)$ -nd Fibonacci number.

14. Let $n \geq 1$ be an integer, and let S_n be the number of ways in which n can be written as a sum of 3's and 4's, such that the order in which the 3's and 4's occur in the sum matters. For example, $S_5 = 0$, because 5 cannot be written as a sum of 3's and 4's. We have $S_{11} = 3$, because

$$11 = 3 + 4 + 4 = 4 + 3 + 4 = 4 + 4 + 3.$$

Which of the following is true for any $n \geq 5$?

- (a) $S_n = 2 \cdot S_{n-1}$
- (b) $S_n = S_{n-1} + S_{n-2}$
- (c) $S_n = S_{n-2} + S_{n-3}$
- (d) $S_n = S_{n-3} + S_{n-4}$

15. Consider the recursive algorithm HELLO, which takes as input an integer $n \geq 0$:

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Algorithm HELLO( $n$ ):
  if  $n = 0$  or  $n = 1$ 
  then print "hello"
  else if  $n$  is even
    then HELLO( $n/2$ )
    else HELLO( $n - 1$ );
        HELLO( $n - 2$ )
    endif;
  endif

```

If we run algorithm HELLO(7), how many times is the word "hello" printed?

- (a) 4
- (b) 5
- (c) 6
- (d) 7

16. You flip a fair coin 15 times. Define the event

$$A = \text{“the result of the first three flips are equal”}.$$

What is $\Pr(A)$?

- (a) $1/2$
- (b) $1/4$
- (c) $1/6$
- (d) $1/8$

17. Let $n \geq 1$ be an integer. A bag contains n red balls and n blue balls. We choose a uniformly random subset of two balls. Define the event

$$A = \text{“this subset consists of one red ball and one blue ball”}.$$

What is $\Pr(A)$?

- (a) $n^2 / \binom{2n}{2}$
- (b) $\binom{2n}{2} / n^2$
- (c) $n^2 / \binom{n}{2}$
- (d) $\binom{n}{2} / n^2$

