

Approximating the Bottleneck Plane Perfect Matching of a Point Set

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Abstract

A bottleneck plane perfect matching of a set of n points in \mathbb{R}^2 is defined to be a perfect non-crossing matching that minimizes the length of the longest edge; the length of this longest edge is known as *bottleneck*. The problem of computing a bottleneck plane perfect matching has been proved to be NP-hard. We present an algorithm that computes a bottleneck plane matching of size at least $\frac{n}{5}$ in $O(n \log^2 n)$ -time. Then we extend our idea toward an $O(n \log n)$ -time approximation algorithm which computes a plane matching of size at least $\frac{2n}{5}$ whose edges have length at most $\sqrt{2} + \sqrt{3}$ times the bottleneck.

Key words: plane matching, bottleneck matching, geometric graph, unit disk graph, approximation algorithm.

1 Introduction

We study the problem of computing a bottleneck non-crossing matching of points in the plane. For a given set P of n points in the plane, where n is even, let $K(P)$ denote the complete Euclidean graph with vertex set P . The *bottleneck plane matching* problem is to find a perfect non-crossing matching of $K(P)$ that minimizes the length of the longest edge. We denote such a matching by M^* . The bottleneck, λ^* , is the length of the longest edge in M^* . The problem of computing M^* has been proved to be NP-hard [1]. Figure 1 in [1] and [3] shows that the longest edge in the minimum weight matching (which is planar) can be unbounded with respect to λ^* . On the other hand the weight of the bottleneck matching can be unbounded with respect to the weight of the minimum weight matching, see Figure 1.



Figure 1: (a) bottleneck matching, (b) minimum weight matching.

Matching and bottleneck matching problems play an important role in graph theory, and thus, they have been studied extensively, e.g., [1, 2, 4, 7, 8, 11, 12]. Self-crossing configurations

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Table 1: Summary of results.

Algorithm	time complexity	bottleneck (λ)	size of matching
Abu-Affash et al. [1]	$O(n^{1.5}\sqrt{\log n})$	$2\sqrt{10}\lambda^*$	$n/2$
Section 4.1	$O(n \log^2 n)$	λ^*	$n/5$
Section 4.2	$O(n \log n)$	$(\sqrt{2} + \sqrt{3})\lambda^*$	$2n/5$

are often undesirable and may even imply an error condition; for example, a potential collision between moving objects, or inconsistency in a layout of a circuit. In particular, non-crossing matchings are especially important in the context of VLSI circuit layouts [9] and operations research.

1.1 Previous Work

It is desirable to compute a perfect matching of a point set in the plane, which is optimal with respect to some criterion such as: (a) *minimum-cost matching* which minimizes the sum of the lengths of all edges; also known as *minimum-weight matching* or *min-sum matching*, and (b) *bottleneck matching* which minimizes the length of the longest edge; also known as *min-max matching* [8]. For the minimum-cost matching, Vaidya [11] presented an $O(n^{2.5} \log^4 n)$ time algorithm, which was improved to $O(n^{1.5} \log^5 n)$ by Varadarajan [12]. As for bottleneck matching, Chang et al. [4] proved that such kind of matching is a subset of 17-RNG (relative neighborhood graph). They presented an algorithm, running in $O(n^{1.5} \sqrt{\log n})$ -time to compute a bottleneck matching of maximum cardinality. The matching computed by their algorithm may be crossing. Efrat and Katz [8] extended the result of Chang et al. [4] to higher dimensions. They proved that a bottleneck matching in any constant dimension can be computed in $O(n^{1.5} \sqrt{\log n})$ -time under the L_∞ -norm.

Note that a plane perfect matching of a point set can be computed in $O(n \log n)$ -time, e.g., by matching the two leftmost points recursively.

Abu-Affash et al. [1] showed that the bottleneck plane perfect matching problem is NP-hard and presented an algorithm that computes a plane perfect matching whose edges have length at most $2\sqrt{10}$ times the bottleneck, i.e., $2\sqrt{10}\lambda^*$. They also showed that this problem does not admit a PTAS (Polynomial Time Approximation Scheme), unless $P=NP$. Carlsson et al. [3] showed that the bottleneck plane perfect matching problem for a Euclidean bipartite complete graph is also NP-hard.

1.2 Our Results

The main results of this paper are summarized in Table 1. We use the unit disk graph as a tool for our approximations. First, we present an $O(n \log n)$ -time algorithm in Section 3, that computes a plane matching of size at least $\frac{n-1}{5}$ in a connected unit disk graph. Then in Section 4.1 we describe how one can use this algorithm to obtain a bottleneck plane matching of size at least $\frac{n}{5}$ with edges of length at most λ^* in $O(n \log^2 n)$ -time. In Section 4.2 we present an $O(n \log n)$ -time approximation algorithm that computes a plane matching of size at least $\frac{2n}{5}$ whose edges have length at most $(\sqrt{2} + \sqrt{3})\lambda^*$. Finally we conclude this paper in Section 5.

2 Preliminaries

Let P denote a set of n points in the plane, where n is even, and let $K(P)$ denote the complete Euclidean graph over P . A *matching*, M , is a subset of edges of $K(P)$ without common vertices. Let $|M|$ denote the *cardinality* of M , which is the number of edges in M . M is a *perfect matching*

if it covers all the vertices of P , i.e., $|M| = \frac{n}{2}$. The *bottleneck* of M is defined as the longest edge in M . We denote its length by λ_M . A *bottleneck perfect matching* is a perfect matching that minimizes the bottleneck length. A *plane matching* is a matching with non-crossing edges. We denote a plane matching by M_{\neq} and a crossing matching by M_{\times} . The *bottleneck plane perfect matching*, M^* , is a perfect plane matching which minimizes the length of the longest edge. Let λ^* denote the length of the bottleneck edge in M^* . In this paper we consider the problem of computing a bottleneck plane matching of P .

The Unit Disk Graph, $UDG(P)$, is defined to have the points of P as its vertices and two vertices p and q are connected by an edge if their Euclidean distance $|pq|$ is at most 1. The *maximum plane matching* of $UDG(P)$ is the maximum cardinality matching of $UDG(P)$, which has no pair of crossing edges.

Lemma 1. *If the maximum plane matching in unit disk graphs can be computed in polynomial time, then the bottleneck plane perfect matching problem for point sets can also be solved in polynomial time.*

Proof. Let $D = \{|pq| : p, q \in P\}$ be the set of all distances determined by pairs of points in P . Note that $\lambda^* \in D$. For each $\lambda \in D$, define the “unit” disk graph $DG(\lambda, P)$, in which two points p and q are connected by an edge if and only if $|pq| \leq \lambda$. Then λ^* is the minimum λ in D such that $DG(\lambda, P)$ has a plane matching of size $\frac{n}{2}$. ■

The Gabriel Graph, $GG(P)$, is defined to have the points of P as its vertices and two vertices p and q are connected by an edge if the disk with diameter pq does not contain any point of $P \setminus \{p, q\}$ in its interior and on its boundary.

Lemma 2. *If the unit disk graph $UDG(P)$ of a point set P is connected, then $UDG(P)$ and $K(P)$ have the same minimum spanning tree.*

Proof. By running Kruskal’s algorithm on $UDG(P)$, we get a minimum spanning tree, say T . All the edges of T have length at most one, and the edges of $K(P)$ which do not belong to $UDG(P)$ all have length greater than one. Hence, T is also a minimum spanning tree of $K(P)$. ■

As a direct consequence of Lemma 2 we have the following corollary:

Corollary 1. *Consider the unit disk graph $UDG(P)$ of a point set P . We can compute the minimum spanning forest of $UDG(P)$, by first computing the minimum spanning tree of P and then removing the edges whose length is more than one.*

Lemma 3. *For each pair of crossing edges (u, v) and (x, y) in $UDG(P)$, the four endpoints $u, v, x,$ and y are in the same component of $UDG(P)$.*

Proof. Note that the quadrilateral Q formed by the end points $u, v, x,$ and y is convex. W.l.o.g. assume that the angle $\angle xuy$ is the largest angle in Q . Clearly $\angle xuy \geq \pi/2$, and hence, in triangle $\triangle xuy$, the angles $\angle yxu$ and $\angle uyx$ are both less than $\pi/2$. Thus, the edges (u, x) and (u, y) are both less than (x, y) . This means that (u, x) and (u, y) are also edges of $UDG(P)$, thus, their four endpoints belong to the same component. ■

As a direct consequence of Lemma 3 we have the following corollary:

Corollary 2. *Any two edges that belong to different components of $UDG(P)$ do not cross.*

Let $MST(P)$ denote the Euclidean minimum spanning tree of P .

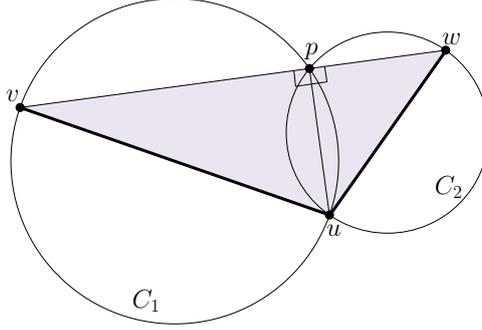


Figure 2: In $MST(P)$, the triangle Δuvw formed by two adjacent edges (u, v) and (u, w) , is empty.

Lemma 4. *If (u, v) and (u, w) are two adjacent edges in $MST(P)$, then the triangle Δuvw has no point of $P \setminus \{u, v, w\}$ inside or on its boundary.*

Proof. If the angle between the line segments \overline{uv} and \overline{uw} is equal to π then clearly there is no other point of P on \overline{uv} and \overline{uw} . So assume that $\angle vuw < \pi$. Refer to Figure 2. Since $MST(P)$ is a subgraph of the Gabriel graph, the circles C_1 and C_2 with diameters \overline{uv} and \overline{uw} are empty. Since $\angle vuw < \pi$, C_1 and C_2 intersect each other at two points, say u and p . Connect u, v and w to p . Since \overline{uv} and \overline{uw} are the diameters of C_1 and C_2 , $\angle upv = \angle wpu = \pi/2$. This means that \overline{vw} is a straight line segment. Since C_1 and C_2 are empty and $\Delta uvw \subset C_1 \cup C_2$, it follows that $\Delta uvw \cap P = \{u, v, w\}$. ■

Corollary 3. *Consider $MST(P)$ and let N_v be the set of neighbors of a vertex v in $MST(P)$. Then the convex hull of N_v contains no point of P except v and the set N_v .*

The shaded area in Figure 3 shows the union of all these convex hulls.

3 Plane Matching in Unit Disk Graphs

In this section we present two approximation algorithms for computing a maximum plane matching in a unit disk graph $UDG(P)$. In Section 3.1 we present a straight-forward $\frac{1}{3}$ -approximation algorithm; it is unclear whether this algorithm runs in polynomial time. For the case when $UDG(P)$ is connected, we present a $\frac{2}{5}$ -approximation algorithm in Section 3.2.

3.1 $\frac{1}{3}$ -approximation algorithm

Given a possibly disconnected unit disk graph $UDG(P)$, we start by computing a (possibly crossing) maximum matching M_\times of $UDG(P)$ using Edmonds algorithm [6]. Then we transform M_\times to another (possibly crossing) matching M'_\times with some properties, and then pick at least one-third of its edges which satisfy the non-crossing property. Consider a pair of crossing edges (p, q) and (r, s) in M_\times , and let c denote the intersection point. If their smallest intersection angle is at most $\pi/3$, we replace these two edges with new ones. For example if $\angle pcr \leq \pi/3$, we replace (p, q) and (r, s) with new edges (p, r) and (q, s) . Since the angle between them is at most $\pi/3$, the new edges are not longer than the older ones, i.e. $\max\{|pr|, |qs|\} \leq \max\{|pq|, |rs|\}$, and hence the new edges belong to the unit disk graph. On the other hand the total length of the new edges is strictly less than the older ones; i.e. $|pr| + |qs| < |pq| + |rs|$. For each pair of intersecting edges in M_\times , with angle at most $\pi/3$, we apply this replacement. We continue this process until we have a matching M'_\times with the property that if two matched edges intersect, each of the angles incident on c is larger than $\pi/3$.

For each edge in M'_\times , consider the counter clockwise angle it makes with the positive x -axis; this angle is in the range $[0, \pi)$. Using these angles, we partition the edges of M'_\times into three subsets, one subset for the angles $[0, \pi/3)$, one subset for the angles $[\pi/3, 2\pi/3)$, and one subset for the angles $[2\pi/3, \pi)$. Observe that edges within one subset are non-crossing. Therefore, if we output the largest subset, we obtain a non-crossing matching of size at least $|M'_\times|/3 = |M_\times|/3$.

Since in each step (each replacement) the total length of the matched edges decreases, the replacement process converges and the algorithm will stop. We do not know whether this process converges in a polynomial number of steps in the size of $UDG(P)$.

3.2 $\frac{2}{5}$ -approximation algorithm for connected unit disk graphs

In this section we assume that the unit disk graph $UDG(P)$ is connected. Monma et al. [10] proved that every set of points in the plane admits a minimum spanning tree of degree at most five which can be computed in $O(n \log n)$ time. By Lemma 2, the same claim holds for $UDG(P)$. Here we present an algorithm which extracts a plane matching M from $MST(P)$. Consider a minimum spanning tree T of $UDG(P)$ with vertices of degree at most five. We define the *skeleton tree*, T' , as the tree obtained from T by removing all its leaves; see Figure 3. Clearly $T' \subseteq T \subseteq UDG(P)$. For clarity we use u and v to refer to the leaves of T and T' respectively. In addition, let v and v' , respectively, refer to the copies of a vertex v in T and T' . In each step, pick an arbitrary leaf $v' \in T'$. By the definition of T' , it is clear that the copy of v' in T , i.e. v , is connected to vertices u_1, \dots, u_k , for some $1 \leq k \leq 4$, which are leaves of T (if T' has one vertex then $k \leq 5$). Pick an arbitrary leaf u_i and add (v, u_i) as a matched pair to M . For the next step we update T by removing v and all its adjacent leaves. We also compute the new skeleton tree and repeat this process. In the last iteration, T' is empty and we may be left with a tree T consisting of one single vertex or one single edge. If T consists of one single vertex, we disregard it, otherwise we add its only edge to M .

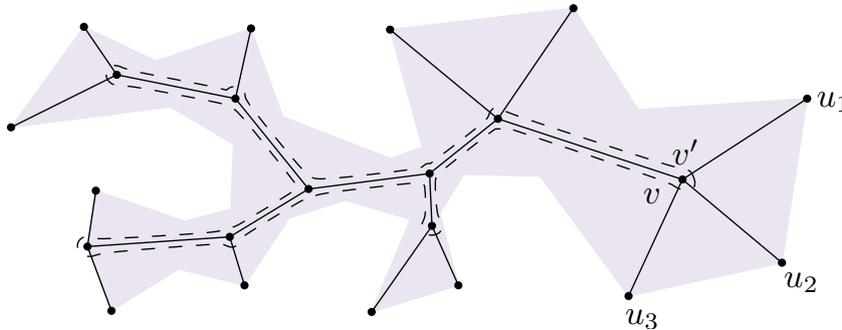


Figure 3: Minimum spanning tree T with union of empty convex hulls. The skeleton tree T' is surrounded by dashed line, and v' is a leaf in T' .

The formal algorithm is given as PLANEMATCHING, which receives a point set P —whose unit disk graph is connected—as input and returns a matching M as output. The function $MST5(P)$ returns a Euclidean minimum spanning tree of P with degree at most five, and the function $Neighbor(v', T')$ returns the only neighbor of leaf v' in T' .

Lemma 5. *Given a set P of n points in the plane such that $UDG(P)$ is connected, algorithm PLANEMATCHING returns a plane matching M of $MST(P)$ of size $|M| \geq \frac{n-1}{5}$. Furthermore, M can be computed in $O(n \log n)$ time.*

Proof. In each iteration an edge $(v, u_i) \in T$ is added to M . Since T is plane, M is also plane and M is a matching of $MST(P)$.

Algorithm 1 PLANEMATCHING(P)

Input: set P of n points in the plane, such that $UDG(P)$ is connected.

Output: plane matching M of $MST(P)$ with $|M| \geq \frac{n-1}{5}$.

```
1:  $M \leftarrow \emptyset$ 
2:  $T \leftarrow \text{MST5}(P)$ 
3:  $T' \leftarrow \text{SkeletonTree}(T)$ 
4: while  $T' \neq \emptyset$  do
5:    $v' \leftarrow$  a leaf of  $T'$ 
6:    $L_v \leftarrow$  set of leaves connected to  $v$  in  $T$ 
7:    $u \leftarrow$  an element of  $L_v$ 
8:    $M \leftarrow M \cup \{(v, u)\}$ 
9:    $T \leftarrow T \setminus (\{v\} \cup L_v)$ 
10:  if  $\text{deg}(\text{Neighbor}(v', T')) = 2$  then
11:     $T' \leftarrow T' \setminus \{v', \text{Neighbor}(v', T')\}$ 
12:  else
13:     $T' \leftarrow T' \setminus \{v'\}$ 
14:  if  $T$  consists of one single edge then
15:     $M \leftarrow M \cup T$ 
16: return  $M$ 
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Line 5 picks $v' \in T'$ which is a leaf, so its analogous vertex $v \in T$ is connected to at least one leaf. In each iteration we select an edge incident to one of the leaves and add it to M , then disregard all other edges connected to v (line 9). So for the next iteration T loses at most five edges. Since T has $n - 1$ edges initially and we add one edge to M out of each five edges of T , we have $|M| \geq \frac{n-1}{5}$.

According to Corollary 1 and by [10], line 2 takes $O(n \log n)$ time. The while-loop is iterated $O(n)$ times and in each iteration, lines 5-13 take constant time. So, the total running time of algorithm PLANEMATCHING is $O(n \log n)$. ■

The size of a maximum matching can be at most $\frac{n}{2}$. Therefore, algorithm PLANEMATCHING computes a matching of size at least $\frac{2(n-1)}{5n}$ times the size of a perfect matching, and hence, when n is large enough, PLANEMATCHING is a $\frac{2}{5}$ -approximation algorithm. On the other hand there are unit disk graphs whose maximum matchings have size $\frac{n-1}{5}$; see Figure 4. In this case PLANEMATCHING returns a maximum matching. In addition, when $UDG(P)$ is a tree or a cycle, PLANEMATCHING returns a maximum matching.

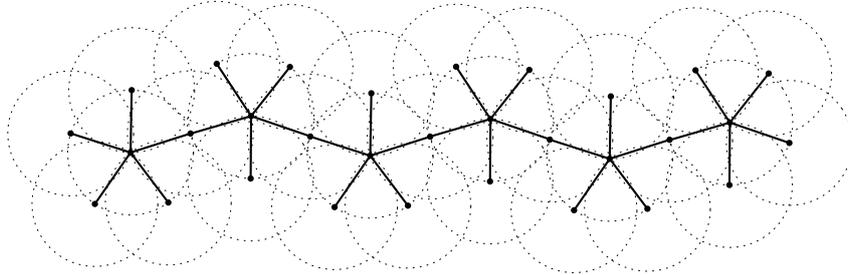


Figure 4: Unit disk graph with all edges of unit length, and maximum matching of size $\frac{n-1}{5}$.

In Section 4.1 we will show how one can use a modified version of algorithm PLANEMATCHING to compute a bottleneck plane matching of size at least $\frac{n}{5}$ with bottleneck length at most λ^* . Recall that λ^* is the length of the bottleneck edge in the bottleneck plane perfect matching

M^* . Section 4.2 extends this idea to an algorithm which computes a plane matching of size at least $\frac{2n}{5}$ with edges of length at most $(\sqrt{2} + \sqrt{3})\lambda^*$.

4 Approximating Bottleneck Plane Perfect Matching

The general approach of our algorithms is to first compute a (possibly crossing) bottleneck perfect matching M_\times of $K(P)$ using the algorithm in [4]. Let λ_{M_\times} denote the length of the bottleneck edge in M_\times . It is obvious that the bottleneck length of any plane perfect matching is not less than λ_{M_\times} . Therefore, $\lambda^* \geq \lambda_{M_\times}$. We consider a “unit” disk graph $DG(\lambda_{M_\times}, P)$ over P , in which there is an edge between two vertices p and q if $|pq| \leq \lambda_{M_\times}$. Note that $DG(\lambda_{M_\times}, P)$ is not necessarily connected. Let G_1, \dots, G_k be the connected components of $DG(\lambda_{M_\times}, P)$. For each component G_i , consider a minimum spanning tree T_i of degree at most five. We show how to extract from T_i a plane matching M_i of proper size and appropriate edge lengths.

Lemma 6. *Each component of $DG(\lambda_{M_\times}, P)$ has an even number of vertices.*

Proof. This follows from the facts that M_\times is a perfect matching and both end points of each edge in M_\times belong to the same component of $DG(\lambda_{M_\times}, P)$. ■

4.1 First Approximation Algorithm

In this section we describe the process of computing a plane matching M of size $|M| \geq \frac{1}{5}n$ with bottleneck length at most λ^* . Consider the minimum spanning trees T_1, \dots, T_k of the k components of $DG(\lambda_{M_\times}, P)$. For $1 \leq i \leq k$, let P_i denote the set of vertices in T_i and n_i denote the number of vertices of P_i . Our approximation algorithm runs in two steps:

Step 1: We start by running algorithm PLANEMATCHING on each of the point sets P_i . Let M_i be the output. Recall that algorithm PLANEMATCHING, from Section 3.2, picks a leaf $v' \in T'_i$, corresponding to a vertex $v \in T_i$, matches it to one of its neighboring leaves in T_i and disregards the other edges connected to v . According to Lemma 5, this gives us a plane matching M_i of size at least $\frac{n_i-1}{5}$. However, we are looking for a matching of size at least $\frac{n_i}{5}$.

The total number of edges of T_i is $n_i - 1$ and in each of the iterations, the algorithm picks one edge out of at most five candidates. If in at least one of the iterations of the while-loop, v has degree at most four (in T_i), then in that iteration algorithm PLANEMATCHING picks one edge out of at most four candidates. Therefore, the size of M_i satisfies

$$|M_i| \geq 1 + \frac{(n_i - 1) - 4}{5} = \frac{n_i}{5}.$$

If in all the iterations of the while-loop, v has degree five, we look at the angles between the consecutive leaves connected to v . Recall that in $MST(P)$ all the angles are greater than or equal to $\pi/3$. If in at least one of the iterations, v is connected to two consecutive leaves u_j and u_{j+1} for $1 \leq j \leq 3$, such that $\angle u_j v u_{j+1} = \pi/3$, we change M_i as follow. Remove from M_i the edge incident to v and add to M_i the edges (u_j, u_{j+1}) and (v, u_s) , where $u_s, s \notin \{j, j+1\}$, is one of the leaves connected to v . Clearly $\triangle v u_j u_{j+1}$ is equilateral and $|u_j u_{j+1}| = |u_j v| = |u_{j+1} v| \leq \lambda_{M_\times}$, and by Lemma 4, (u_j, u_{j+1}) does not cross other edges. In this case, the size of M_i satisfies

$$|M_i| = 2 + \frac{(n_i - 1) - 5}{5} = \frac{n_i + 4}{5} \geq \frac{n_i}{5}.$$

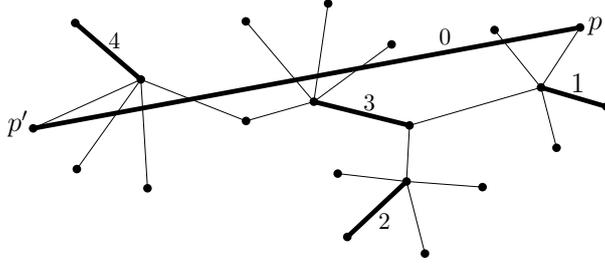


Figure 5: Resulted matching M_i by modified PLANEMATCHING. The numbers show the order in which the edges (bold edges) are added to M_i .

Step 2: In this step we deal with the case that in all the iterations of the while-loop, v has degree five and the angle between any pair of consecutive leaves connected to v is greater than $\pi/3$. Recall that M_\times is a perfect matching and both end points of each edge in M_\times belong to the same T_i .

Lemma 7. *In Step 2, at least two leaves of T_i are matched in M_\times .*

Proof. Let m_i and m'_i denote the number of external (leaves) and internal nodes of T_i , respectively. Clearly m'_i is equal to the number of vertices of T'_i and $m_i + m'_i = n_i$. Consider the reverse process in PLANEMATCHING. Start with a 5-star tree t_i , i.e. $t_i = K_{1,5}$, and in each iteration append a new $K_{1,5}$ to t_i until $t_i = T_i$. In the first step $m_i = 5$ and $m'_i = 1$. In each iteration a leaf of the appended $K_{1,5}$ is identified with a leaf of t_i ; the resulting vertex becomes an internal node. On the other hand, the “center” of the new $K_{1,5}$ becomes an internal node of t_i and its other four neighbors become leaves of t_i . So in each iteration, the number of leaves m_i increases by three, and the number of internal nodes m'_i increases by two. Hence, in all iterations (including the first step) we have $m_i \geq m'_i + 4$.

Again consider M_\times . In the worst case if all m'_i internal vertices of T_i are matched to leaves, we still have four leaves which have to be matched together. ■

According to Lemma 7 there is an edge $(p, p') \in M_\times$ where p and p' are leaves in T_i . We can find p and p' for all T_i 's by checking all the edges of M_\times once. We remove all the edges of M_i and initialize $M = \{(p, p')\}$. Again we run a modified version of PLANEMATCHING in such a way that in each iteration, in line 7 it selects the leaf u_i adjacent to v such that (v, u_i) is not intersected by (p, p') . In each iteration v has degree five and is connected to at least four leaf edges with angles greater than $\pi/3$. Thus, (p, p') can intersect at most three of the leaf edges and such kind of (v, u_i) exists. See Figure 5. In this case, M_i has size

$$|M_i| = 1 + \frac{n_i - 1}{5} = \frac{n_i + 4}{5} \geq \frac{n_i}{5}.$$

We run the above algorithm on each T_i and for each of them we compute a plane matching M_i . The final matching of point set P will be $M = \bigcup_{i=1}^k M_i$.

Theorem 1. *Let P be a set of n points in the plane, where n is even, and let λ^* be the minimum bottleneck length of any plane perfect matching of P . In $O(n^{1.5}\sqrt{\log n})$ time, a plane matching of P of size at least $\frac{n}{5}$ can be computed, whose bottleneck length is at most λ^* .*

Proof. Proof of edge length: Let λ_{M_\times} be the length of the longest edge in M_\times and consider a component G_i of $DG(\lambda_{M_\times}, P)$. All the selected edges in Steps 1 and 2 belong to T_i except (u_j, u_{j+1}) and (p, p') . T_i is a subgraph of G_i , and the edge (u_j, u_{j+1}) belongs to G_i , and the edge (p, p') belongs to M_\times (which belongs to G_i as well). So all the selected edges belong to G_i , and $\lambda_{M_i} \leq \lambda_{M_\times}$. Since $\lambda_{M_\times} \leq \lambda^*$, we have $\lambda_{M_i} \leq \lambda_{M_\times} \leq \lambda^*$ for all i , $1 \leq i \leq k$.

Proof of planarity: The edges of M_i belong to the minimum spanning forest of $DG(\lambda_{M_\times}, P)$ which is plane, except (u_j, u_{j+1}) and (p, p') . According to Corollary 2 and Lemma 4 the edge (u_j, u_{j+1}) does not cross the edges of the minimum spanning forest. In Step 2 we select edges of T_i in such a way that avoid (p, p') . Note that (p, p') belongs to the component G_i and by Corollary 2 it does not cross any edge of the other components of $DG(\lambda_{M_\times}, P)$. So M is plane.

Proof of matching size: Since $M = M_1 \cup \dots \cup M_k$, and for each $1 \leq i \leq k$, $|M_i| \geq \frac{n_i}{5}$, hence

$$|M| \geq \sum_{i=1}^k |M_i| \geq \sum_{i=1}^k \frac{n_i}{5} = \frac{n}{5}.$$

Proof of complexity: The initial matching M_\times can be computed in time $O(n^{1.5}\sqrt{\log n})$ by using the algorithm of Chang et al. [4]. By Lemma 5 algorithm PLANEMATCHING runs in $O(n \log n)$ time. In Step 1 we spend constant time for checking the angles and the number of leaves connected to v during the while-loop. In Step 2, the matched leaves p and p' can be computed in $O(n)$ time for all T_i 's by checking all the edges of M_\times before running the algorithm again. So the modified PLANEMATCHING still runs in $O(n \log n)$ time, and the total running time of our method is $O(n^{1.5}\sqrt{\log n})$. ■

Since the running time of the algorithm is bounded by the time of computing the initial bottleneck matching M_\times , any improvement in computing M_\times leads to a faster algorithm for computing a plane matching M . In the next section we improve the running time.

4.1.1 Improving the Running Time

In this section we present an algorithm that improves the running time to $O(n \log^2 n)$. We first compute a forest F , such that each edge in F is of length at most λ^* and, for each tree $T \in F$, we have a leaf $p \in T$ and a point $p' \in T$ such that $|pp'| \leq \lambda^*$. Once we have this forest F , we apply Step 1 and Step 2 on F to obtain the matching M as in the previous section.

Let $MST(P)$ be a (five-degree) minimum spanning tree of P . Let $F_\lambda = \{T_1, T_2, \dots, T_k\}$ be the forest obtained from $MST(P)$ by removing all the edges whose length is greater than λ , i.e., $F_\lambda = \{e \in MST(P) : |e| \leq \lambda\}$. For a point $p \in P$, let $cl(p, P)$ be the point in P that is closest to p .

Lemma 8. *For all $T \in F_{\lambda^*}$, it holds that*

(i) *the number of points in T is even, and*

(ii) *for each two leaves $p, q \in T$ that are incident to the same node v in T , let $p' = cl(p, P \setminus \{v\})$, let $q' = cl(q, P \setminus \{v\})$, and assume that $|pp'| \leq |qq'|$. Then, $\lambda^* \geq |pp'|$ and p' belongs to T .*

Proof. (i) Suppose that T has odd number of points. Thus in M^* one of the points in T should be matched to a point in a tree $T' \neq T$ by an edge e . Since $e \notin F_{\lambda^*}$, we have $|e| > \lambda^*$, which contradicts that λ^* is the minimum bottleneck. (ii) Note that v is the closest point to both p and q . In M^* , at most one of p and q can be matched to v , and the other one must be matched to a point which is at least as far as its second closest point. Thus, λ^* is at least $|pp'|$. The distance between any two trees in F_{λ^*} is greater than λ^* . Now if p' is not in T , then in any bottleneck perfect matching, either p or q is matched to a point of distance greater than λ^* , which contradicts that λ^* is the minimum bottleneck. ■

Let $E = (e_1, e_2, \dots, e_{n-1})$ be the edges of $MST(P)$ in sorted order of their lengths. Our algorithm performs a binary search on E , and for each considered edge e_i , it uses Algorithm 2 to decide whether $\lambda < \lambda^*$, where $\lambda = |e_i|$. The algorithm constructs the forest F_λ , and for each

tree T in F_λ , it picks two leaves p and q from T and finds their second closest points p' and q' . Assume w.l.o.g. that $|pp'| \leq |qq'|$. Then, the algorithm returns FALSE if p' does not belong to T . By Lemma 8, if the algorithm returns FALSE, then we know that $\lambda < \lambda^*$.

Let e_j be the shortest edge in $MST(P)$, for which Algorithm 2 does not return FALSE. This means that Algorithm 2 returns FALSE for e_{j-1} , and there is a tree T in $F_{|e_{j-1}|}$ and a leaf p in T , such that $|pp'| \geq |e_j|$. Thus $|e_j| \leq \lambda^*$ and for each tree T in the forest $F_{|e_j|}$, we stored a leaf p of T and a point $p' \in T$, such that $|pp'| \leq \lambda^*$. Since each tree in $F_{|e_j|}$ is a subtree of $MST(P)$, $F_{|e_j|}$ is planar and each tree in $F_{|e_j|}$ is of degree at most five.

Algorithm 2 COMPARETOOPT(λ)

```

1: compute  $F_\lambda$ 
2:  $L \leftarrow$  empty list
3: for each  $T \in F_\lambda$  do
4:   if  $T$  has an odd number of points then
5:     return FALSE
6:   if there exist two leaves  $p$  and  $q$  incident to a node  $v \in T$  then
7:      $p' \leftarrow cl(p, P \setminus \{v\})$ 
8:      $q' \leftarrow cl(q, P \setminus \{v\})$ 
9:     if  $|pp'| \leq |qq'|$  then
10:      if  $p'$  does not belong to  $T$  then
11:        return FALSE
12:      else
13:        add the triple  $(p, p', T)$  to  $L$ 
14:      else
15:        if  $q'$  does not belong to  $T$  then
16:          return FALSE
17:        else
18:          add the triple  $(q, q', T)$  to  $L$ 
19: return  $L$ 

```

Now we can apply Step 1 and Step 2 on $F_{|e_j|}$ as in the previous section. Note that in Step 2, for each tree T_i we have a pair (p, p') (or (q, q')) in the list L which can be matched. In Step 2, in each iteration v has degree five, thus, p' should be a vertex of degree two or a leaf in T_i . If p' is a leaf we run the modified version of PLANEMATCHING as in the previous section. If p' has degree two, we remove all the edges of M_i and initialize $M_i = (p, p')$. Then remove p' from T_i and run PLANEMATCHING on the resulted subtrees. Finally, set $M = \bigcup_{i=1}^k M_i$.

Lemma 9. *The matching M is planar.*

Proof. Consider two edges $e = (p, p')$ and $e' = (q, q')$ in M . We distinguish between four cases:

1. $e \in F_{|e_j|}$ and $e' \in F_{|e_j|}$. In this case, both e and e' belong to $MST(P)$ and hence they do not cross each other.
2. $e \notin F_{|e_j|}$ and $e' \in F_{|e_j|}$. If e and e' cross each other, then this contradicts the selection of (q, q') in Step 2 (which prevents (p, p')).
3. $e \in F_{|e_j|}$ and $e' \notin F_{|e_j|}$. It leads to a contradiction as in the previous case.
4. $e \notin F_{|e_j|}$ and $e' \notin F_{|e_j|}$. If e and e' cross each other, then either $\min\{|pq|, |pq'|\} < |pp'|$ or $\min\{|qp|, |qp'|\} < |qq'|$, which contradicts the selection of p' or q' . Note that p cannot be the second closest point to q , because p and q are in different trees.

■

Lemma 10. *The matching M can be computed in $O(n \log^2 n)$ time.*

Proof. Computing $MST(P)$ and sorting its edges take $O(n \log n)$ [5]. Since we performed a binary search on the edges of $MST(P)$, we need $\log n$ iterations. In each iteration, for an edge e_i , we compute the forest $F_{|e_i|}$ in $O(n)$ and the number of the trees in the forest can be $O(n)$ in the worst case. We compute in advance the second order Voronoi diagram of the points together with a corresponding point location data structure, in $O(n \log n)$ [5]. For each tree in the forest, we perform a point location query to find the closest points p' and q' , which takes $O(\log n)$ for each query. Therefore the total running time is $O(n \log^2 n)$. ■

Theorem 2. *Let P be a set of n points in the plane, where n is even, and let λ^* be the minimum bottleneck length of any plane perfect matching of P . In $O(n \log^2 n)$ time, a plane matching of P of size at least $\frac{n}{5}$ can be computed, whose bottleneck length is at most λ^* .*

4.2 Second Approximation Algorithm

In this section we present another approximation algorithm which gives a plane matching M of size $|M| \geq \frac{2}{5}n$ with bottleneck length $\lambda_M \leq (\sqrt{2} + \sqrt{3})\lambda^*$. Let $DT(P)$ denote the Delaunay triangulation of P . Let the edges of $DT(P)$ be, in sorted order of their lengths, e_1, e_2, \dots . Initialize a forest F consisting of n trees, each one being a single node for one point of P . Run Kruskal's algorithm on the edges of $DT(P)$ and terminate as soon as every tree in F has an even number of nodes. Let e_l be the last edge that is added by Kruskal's algorithm. Observe that e_l is the longest edge in F . Denote the trees in F by T_1, \dots, T_k and for $1 \leq i \leq k$, let P_i be the vertex set of T_i and let $n_i = |P_i|$.

Lemma 11. $\lambda^* \geq |e_l|$.

Proof. Let i be such that e_l is an edge in T_i . Let T'_i and T''_i be the trees obtained by removing e_l from T_i . Let P'_i be the vertex set of T'_i . Then $|e_l| = \min\{|pq| : p \in P'_i, q \in P \setminus P'_i\}$. Consider the optimal matching M^* with bottleneck length λ^* . Since e_l is the last edge added, P'_i has odd size. The matching M^* contains an edge joining a point in P'_i with a point in $P \setminus P'_i$. This edge has length at least $|e_l|$. ■

By Lemma 11 the length of the longest edge in F is at most λ^* . For each $T_i \in F$, where $1 \leq i \leq k$, our algorithm will compute a plane matching M_i of P_i of size at least $\frac{2n_i}{5}$ with edges of length at most $(\sqrt{2} + \sqrt{3})\lambda^*$ and returns $\bigcup_{i=1}^k M_i$. To describe the algorithm for tree T_i on vertex set P_i , we will write P, T, n, M instead of P_i, T_i, n_i, M_i , respectively. Thus, P is a set of n points, where n is even, and T is a minimum spanning tree of P_i .

Consider the minimum spanning tree T of P having degree at most five, and let T' be the skeleton tree of T . Suppose that T' has at least two vertices. We will use the following notation. Let v' be a leaf in T' , and let w' be the neighbor of v' . Recall that v' and w' are copies of vertices v and w in T . In T , we consider the clockwise ordering of the neighbors of v . Let this ordering be w, u_1, u_2, \dots, u_k for some $1 \leq k \leq 4$. Clearly u_1, \dots, u_k are leaves in T . Consider two leaves u_i and u_j where $i < j$. We define $cw(u_i v u_j)$ as the clockwise angle from $\overline{vu_i}$ to $\overline{vu_j}$. We say that the leaf v' (or its copy v) is an *anchor* if $k = 2$ and $cw(u_1 v u_2) \geq \pi$. See Figure 6.

Now we describe how one can iteratively compute a plane matching of proper size with bounded-length edges from T . We start with an empty matching M . Each iteration consists of two steps, during which we add edges to M . As we prove later, the output is a plane matching of P of size at least $\frac{2}{5}n$ with bottleneck at most $(\sqrt{2} + \sqrt{3})\lambda^*$.

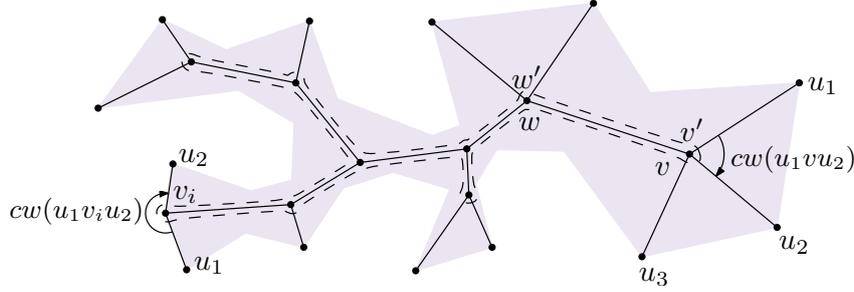


Figure 6: The vertices around v are sorted clockwise, and v_i is an anchor vertex.

4.2.1 Step 1

We keep applying the following process as long as T has more than six vertices and T' has some non-anchor leaf. Note that T' has at least two vertices.

Take a non-anchor leaf v' in T' and according to the number k of leaves connected to v in T do the following:

$k = 1$ add (v, u_1) to M , and set $T = T \setminus \{v, u_1\}$.

$k = 2$ since v' is not an anchor, $cw(u_1 v u_2) < \pi$. By Lemma 4 the triangle $\triangle u_1 v u_2$ is empty. We add (u_1, u_2) to M , and set $T = T \setminus \{u_1, u_2\}$.

$k = 3$ in this case v has degree four and at least one of $cw(u_1 v u_2)$ and $cw(u_2 v u_3)$ is less than π . W.l.o.g. suppose that $cw(u_1 v u_2) < \pi$. By Lemma 4 the triangle $\triangle u_1 v u_2$ is empty. Add (u_1, u_2) to M and set $T = T \setminus \{u_1, u_2\}$.

$k = 4$ this case is handled similarly as the case $k = 3$.

At the end of Step 1, T has at most six vertices or all the leaves of T' are anchors. In the former case, we add edges to M as will be described in Section 4.2.3 and after which the algorithm terminates. In the latter case we go to Step 2.

4.2.2 Step 2

In this step we deal with the case that T has more than six vertices and all the leaves of T' are anchors. We define the *second level skeleton tree* T'' to be the skeleton tree of T' . In other words, T'' is the tree which is obtained from T' by removing all the leaves. For clarity we use w to refer to a leaf of T'' , and we use w, w' , and w'' , respectively, to refer to the copies of vertex w in T, T' , and T'' . For now suppose that T'' has at least two vertices. Consider a leaf w'' and its neighbor y'' in T'' . Note that in T , w is connected to y , to at least one anchor, and possibly to some leaves of T . After Step 1, the copy of w'' in T' , i.e. w' , is connected to anchors v'_1, \dots, v'_k in T' (or v_1, \dots, v_k in T) for some $1 \leq k \leq 4$, and connected to at most $4 - k$ leaves of T . In T , we consider the clockwise ordering of the non-leaf neighbors of w . Let this ordering be y, v_1, v_2, \dots, v_k . We denote the pair of leaves connected to anchor v_i by a_i and b_i in clockwise order around v_i ; see Figure 7.

In this step we pick an arbitrary leaf $w'' \in T''$ and according to the number of anchors incident to w'' , i.e. k , we add edges to M . Since $1 \leq k \leq 4$, four cases occur and we discuss each case separately. Before that, we state some lemmas.

Lemma 12. *Let w'' be a leaf in T'' . Consider an anchor v_i which is adjacent to w in T . For any neighbor x of w for which $x \neq v_i$, if $cw(v_i w x) \leq \pi/2$ (resp. $cw(x w v_i) \leq \pi/2$), the polygon $P_1 = \{v_i, x, w, a_i\}$ (resp. $P_2 = \{v_i, b_i, w, x\}$) is convex and empty.*

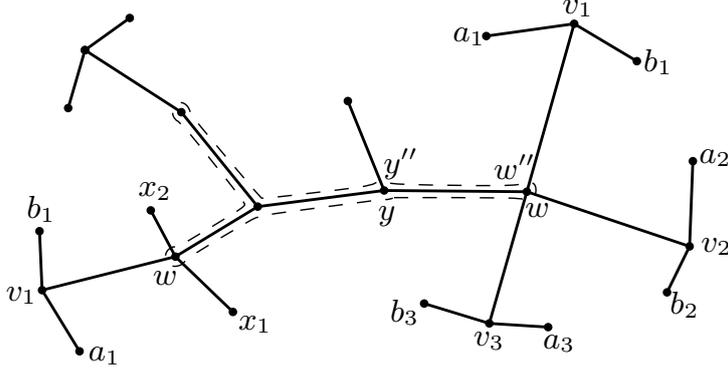


Figure 7: Second level skeleton tree T'' is surrounded by dashed line. v_i 's are ordered clockwise around leaf vertex w'' , as well as x_i 's. a_i and b_i are ordered clockwise around v_i .

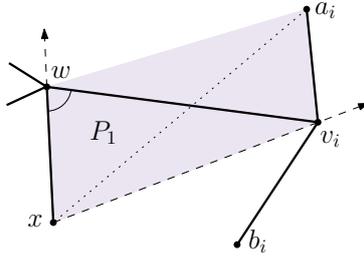


Figure 8: Illustrating Lemma 12.

Proof. We prove the case when $cw(v_iwx) \leq \pi/2$; see Figure 8. The proof for the second case is symmetric. To prove the convexity of P_1 we show that the diagonals $\overline{v_iw}$ and $\overline{a_ix}$ of P_1 intersect each other. To show the intersection we argue that a_i lies to the left of $\overrightarrow{xv_i}$ and to the right of \overrightarrow{xw} .

Consider $\overrightarrow{xv_i}$. According to Lemma 4, triangle Δv_ixw is empty so b_i lies to the right of $\overrightarrow{xv_i}$. On the other hand, v_i is an anchor, so $cw(a_iv_ib_i) \geq \pi$, and hence a_i lies to the left of $\overrightarrow{xv_i}$. Now consider \overrightarrow{xw} . For the sake of contradiction, suppose that a_i is to the left of \overrightarrow{xw} . Since $cw(v_iwx) \leq \pi/2$, the angle $cw(a_iwv_i)$ is greater than $\pi/2$. This means that $\overline{v_ia_i}$ is the largest side of $\Delta a_iv_iv_i$, which contradicts that $\overline{v_ia_i}$ is an edge of $MST(P)$. So a_i lies to the right of \overrightarrow{xw} . Therefore, $\overline{v_iw}$ intersects $\overline{a_ix}$ and P_1 is convex. P_1 is empty because by Lemma 4, the triangles Δv_ixw and Δv_iwa_i are empty. ■

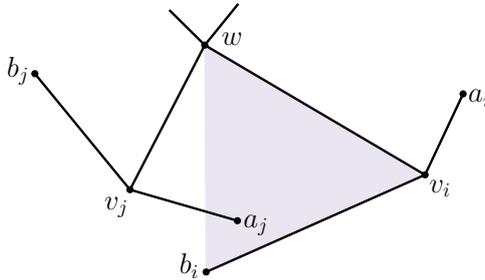


Figure 9: Proof of Lemma 13.

Lemma 13. Let w'' be a leaf in T'' and consider the clockwise sequence v_1, \dots, v_k of anchors that are incident on w'' . The sequence of vertices $a_1, v_1, b_1, \dots, a_k, v_k, b_k$ are angularly sorted in clockwise order around w'' .

Proof. Using contradiction, consider two vertices v_i and v_j , and assume that v_i comes before v_j and a_j comes before b_i in the clockwise order; see Figure 9. Either a_j is in $\triangle wv_i b_i$ or b_i is in $\triangle wv_j a_j$. However, by Lemma 4, neither of these two cases can happen. ■

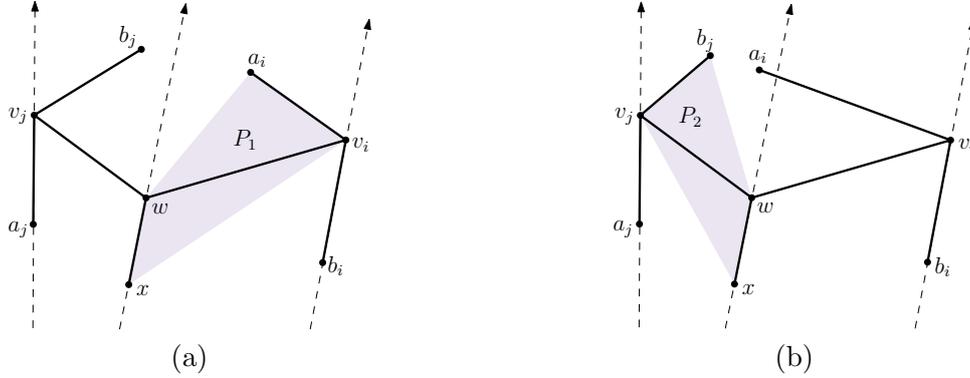


Figure 10: Illustrating Lemma 14.

Lemma 14. *Let w'' be a leaf in T'' and consider the clockwise sequence v_1, \dots, v_k of anchors that are adjacent to w . Let $1 \leq i < j \leq k$ and let x be a neighbor of w for which $x \neq v_i$ and $x \neq v_j$.*

1. *If x is between v_i and v_j in the clockwise order:*
 - (a) *if a_i is to the right of \overrightarrow{xw} , then $P_1 = \{x, w, a_i, v_i\}$ is convex and empty.*
 - (b) *if a_i is not to the right of \overrightarrow{xw} , then $P_2 = \{w, x, v_j, b_j\}$ is convex and empty.*
2. *If v_j is between v_i and x , or v_i is between x and v_j in the clockwise order:*
 - (a) *if b_i is to the left of \overrightarrow{xw} , then $P_3 = \{w, x, v_i, b_i\}$ is convex and empty.*
 - (b) *if b_i is not to the left of \overrightarrow{xw} , then $P_4 = \{x, w, a_j, v_j\}$ is convex and empty.*

Proof. We only prove the first case, the proof for the second case is symmetric. Thus, we assume that x is between v_i and v_j in the clockwise order. First assume that a_i is to the right of \overrightarrow{xw} . See Figure 10(a). Consider $\overrightarrow{b_i v_i}$. Since v_i is an anchor, a_i cannot be to the right of $\overrightarrow{b_i v_i}$, and according to Lemma 4, x cannot be to the right of $\overrightarrow{b_i v_i}$. For the same reasons, both the vertices b_j and x cannot be to the left of $\overrightarrow{a_j v_j}$. Now consider \overrightarrow{xw} . By assumption, a_i is to the right of \overrightarrow{xw} . Therefore $\overrightarrow{xa_i}$ intersects $\overrightarrow{wv_i}$ and hence P_1 is convex.

Now assume that a_i is not to the right of \overrightarrow{xw} ; see Figure 10(b). By Lemma 13, b_j is to the left of \overrightarrow{xw} . Therefore, $\overrightarrow{xb_j}$ intersects $\overrightarrow{wv_j}$ and hence P_2 is convex. The emptiness of the polygons follows directly from Lemma 4. ■

Lemma 15. *Let w'' be a leaf in T'' and consider the clockwise sequence v_1, \dots, v_k of anchors that are incident on w . If $4 \leq k \leq 5$, then at least one of the pentagons $P_1 = \{a_1, v_1, v_2, b_2, w\}$ and $P_2 = \{a_{k-1}, v_{k-1}, v_k, b_k, w\}$ is convex and empty.*

Proof. We prove the case when $k = 4$; the proof for $k = 5$ is analogous. By Lemma 13, a_1 comes before b_4 in the clockwise order. Consider $\overrightarrow{a_1 w}$; see Figure 11. Let $l(a_1, b_2)$ denote the line through a_1 and b_2 . If b_2 is to the left of $\overrightarrow{a_1 w}$, then $l(a_1, b_2)$ separates w from v_1 and v_2 . Otherwise, b_2 is to the right of $\overrightarrow{a_1 w}$. Now consider $\overrightarrow{b_4 w}$. If a_3 is to the right of $\overrightarrow{b_4 w}$, then $l(a_3, b_4)$

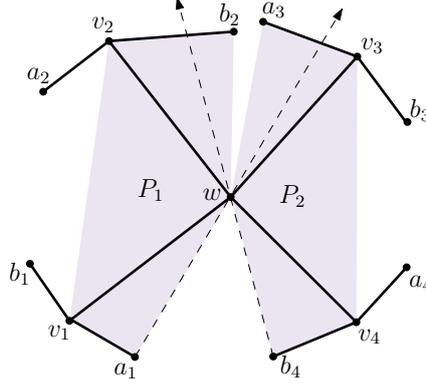


Figure 11: Illustrating Lemma 15.

separates w from v_3 and v_4 . The remaining case, i.e., when a_3 is to the left of $\overrightarrow{b_4w}$, cannot happen by Lemma 13 (because, otherwise, a_3 would come before b_2 in the clockwise order).

Thus, we have shown that (i) $l(a_1, b_2)$ separates w from v_1 and v_2 or (ii) $l(a_3, b_4)$ separates w from v_3 and v_4 . Assume w.l.o.g. that (i) holds. Now to prove the convexity of P_1 we show that all internal angles of P_1 are less than π . Since v_1 is an anchor, $cw(b_1v_1a_1) \leq \pi$. By Lemma 4, b_1 is to the left of $\overrightarrow{v_1v_2}$. Therefore, $cw(v_2v_1a_1) < cw(b_1v_1a_1) \leq \pi$. On the other hand $cw(wv_1a_1) \geq \pi/3$, so in Δa_1v_1w , $cw(v_1a_1w) \leq 2\pi/3$. By a similar analysis $cw(b_2v_2v_1)$ and $cw(wb_2v_2)$ are less than π . In addition, $cw(a_1wb_2)$ in Δa_1wb_2 is less than π . Thus, P_1 is convex. Its emptiness is assured from the emptiness of the triangles Δv_2wb_2 , Δv_1wa_1 and Δv_1wv_2 . ■

Now we are ready to present the details of Step 2. Recall that T has more than six vertices and all leaves of T' are anchors. In this case T'' has at least one vertex. If T'' has exactly one vertex, we add edges to M as will be described in Section 4.2.3 and after which the algorithm terminates. Assume that T'' has at least two vertices. Pick a leaf w'' in T'' . As before, let v_1, \dots, v_k where $1 \leq k \leq 4$ be the clockwise order of the anchors connected to w . Let x_1, \dots, x_ℓ where $\ell \leq 4 - k \leq 3$ be the clockwise order of the leaves of T connected to w ; see Figure 7. This means that $deg(w) = 1 + k + \ell$, where $k + \ell \leq 4$. Now we describe different configurations that may appear at w , according to k and ℓ .

Case 1: assume $k = 1$. If $\ell = 0$ or 1 , add (a_1, v_1) and (b_1, w) to M and set $T = T \setminus \{a_1, b_1, v_1, w\}$. If $\ell = 1$ remove x_1 from T as well. If $\ell = 2$, consider α_i as the acute angle between segments $\overline{wv_1}$ and $\overline{wx_i}$. W.l.o.g. assume $\alpha_1 = \min(\alpha_1, \alpha_2)$. Two cases may arise: (i) $\alpha_1 \leq \pi/2$, (ii) $\alpha_1 > \pi/2$. If (i) holds, w.l.o.g. assume that x_1 comes before v_1 in clockwise order around w . According to Lemma 12 polygon $P = \{v_1, b_1, w, x_1\}$ is convex and empty. So we add (v_1, a_1) , (b_1, x_1) and (x_2, w) to M . Other cases are handled in a similar way. If (ii) holds, according to Lemma 4 triangle Δx_1wx_2 is empty. So we add (v_1, a_1) , (b_1, w) and (x_1, x_2) to M . In both cases set $T = T \setminus \{a_1, b_1, v_1, x_1, x_2, w\}$. If $\ell = 3$, remove x_3 from T and handle the rest as $\ell = 2$.

Case 2: assume $k = 2$. If $\ell = 0$, we add (v_1, a_1) , (b_1, w) , and (v_2, a_2) to M . If $\ell = 1$ suppose that x_1 comes before v_1 in clockwise ordering. According to Lemma 14 one of polygons P_3 and P_4 is empty; suppose it be $P_3 = \{w, x_1, v_1, b_1\}$ (where $i = 1$ and $j = 2$ in Lemma 14). Thus we add (v_1, a_1) , (b_1, x_1) , (v_2, a_2) and (b_2, w) to M . In both cases set $T = T \setminus \{a_1, b_1, v_1, a_2, b_2, v_2, w\}$ and if $\ell = 1$ remove x_1 from T as well. If $\ell = 2$, remove x_2 from T and handle the rest as $\ell = 1$.

Case 3: assume $k = 3$. If $\ell = 0$ then set $M = M \cup \{(v_1, a_1), (b_1, w), (v_2, a_2), (v_3, a_3)\}$. If $\ell = 1$, consider β_i as the acute angle between segments $\overline{wx_1}$ and $\overline{wv_i}$. W.l.o.g. assume β_2 has minimum value among all β_i 's and x_1 comes after v_2 in clockwise order. According to Lemma 14 one of polygons P_1 and P_2 is empty; suppose it be $P_1 = \{w, x_1, v_2, a_2\}$ (where $i = 2$ and $j = 3$ in Lemma 14). Thus we set $M = M \cup \{(v_2, b_2), (a_2, x_1), (v_1, a_1), (b_1, w), (v_3, a_3)\}$. In both cases set $T = T \setminus \{a_1, b_1, v_1, a_2, b_2, v_2, a_3, b_3, v_3, w\}$ and if $\ell = 1$ remove x_1 from T as well. Other cases can be handled similarly.

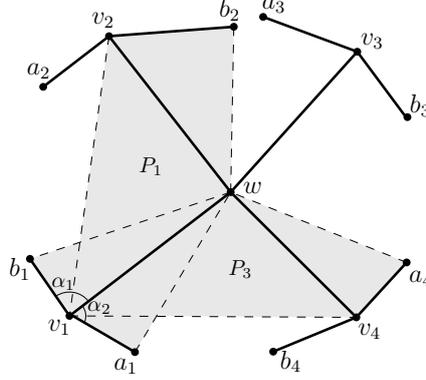


Figure 12: The vertex v_1 is shared between P_1 and P_3 .

Case 4: assume $k = 4$. According to Lemma 15 one of $P_1 = \{a_1, v_1, v_2, b_2, w\}$ and $P_2 = \{a_3, v_3, v_4, b_4, w\}$ is convex and empty. Again by Lemma 15 one of $P_3 = \{a_4, v_4, v_1, b_1, w\}$ and $P_4 = \{a_2, v_2, v_3, b_3, w\}$ is also convex and empty. Without loss of generality assume that P_1 and P_3 are empty. See Figure 12. Clearly, these two polygons share a vertex (v_1 in Figure 12). Let $\alpha_1 = \angle cw(b_1, v_1, w)$ which is contained in P_3 and $\alpha_2 = \angle cw(w, v_1, a_1)$ which is contained in P_1 . We pick one of the polygons P_1 and P_3 which minimizes $\alpha_i, i = 1, 2$. Let P_1 be that polygon. So we set $M = M \cup \{(v_1, b_1), (v_2, a_2), (a_1, b_2), (v_3, a_3), (b_3, w), (v_4, a_4)\}$ and set $T = T \setminus \{a_1, b_1, v_1, a_2, b_2, v_2, a_3, b_3, v_3, a_4, b_4, v_4, w\}$.

This concludes Step 2. Go back to Step 1.

4.2.3 Base Cases

In this section we describe the base cases of our algorithm. As mentioned in Steps 1 and 2, we may have two base cases: (a) T has at most $t \leq 6$ vertices, (b) T'' has only one vertex.

(a) $t \leq 6$ Suppose that T has at most six vertices.

$t = 2$ it can happen only if $t = n = 2$, and we add the only edge to M .

$t = 4, 5$ in this case we match four vertices. If $t = 4$, T could be a star or a path of length three, and in both cases we match all the vertices. If $t = 5$, remove one of the leaves and match other four vertices.

$t = 6$ in this case we match all the vertices. If one of the leaves connected to a vertex of degree two, we match those two vertices and handle the rest as the case when $t = 4$, otherwise, each leaf of T is connected to a vertex of degree more than two, and hence T' has at most two vertices. Figure 13(a) shows the solution for the case when T' has only one vertex and T is a star; note that at least two angles are less than π . Now consider the case when T' has two vertices, v_1 and v_2 , which have degree three in T . Figure 13(b) shows

the solution for the case when neither v_1 nor v_2 is an anchor. Figure 13(c) shows the solution for the case when v_1 is an anchor but v_2 is not. Figure 13(d) shows the solution for the case when both v_1 and v_2 are anchors. Since v_2 is an anchor in Figure 13(d), at least one of $cw(b_2v_2v_1)$ and $cw(v_1v_2a_2)$ is less than or equal to $\pi/2$. W.l.o.g. assume $cw(v_1v_2a_2) \leq \pi/2$. By Lemma 12 polygon $P = \{v_1, a_2, v_2, a_1\}$ is convex and empty. We add (v_1, b_1) , (v_2, b_2) , and (a_1, a_2) to M .

$t = 1, 3$ this case could not happen. Initially $t = n$ is even. Consider Step 1; before each iteration t is bigger than six and during the iteration two vertices are removed from T . So, at the end of Step 1, t is at least five. Now consider Step 2; before each iteration T'' has at least two vertices and during the iteration at most one vertex is removed from T'' . So, at the end of Step 2, T'' has at least one vertex that is connected to at least one anchor. This means that t is at least four. Thus, t could never be one or three before and during the execution of the algorithm.

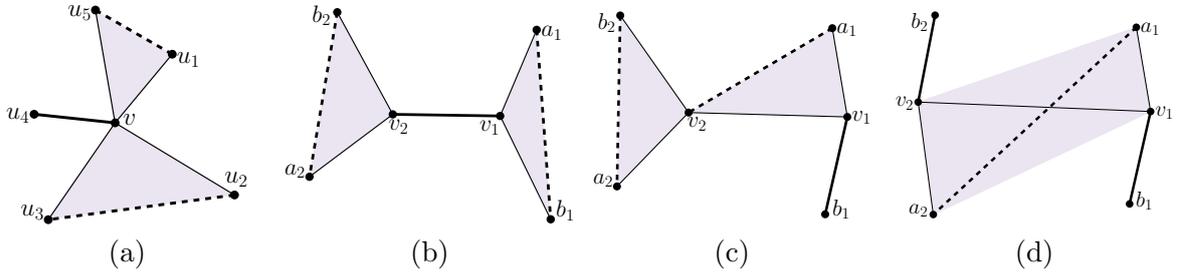


Figure 13: The bold (solid and dashed) edges are added to M and all vertices are matched. (a) a star, (b) no anchor, (c) one anchor, and (d) two anchors.

(b) T'' has one vertex In this case, the only vertex $w'' \in T''$ is connected to at least two anchors, otherwise w'' would have been matched in Step 1. So we consider different cases when w is connected to k , $2 \leq k \leq 5$ anchors and $\ell \leq 5 - k$ leaves of T :

$k = 2$ if $\ell = 0, 1, 2$ we handle it as Case 2 in Step 2. If $\ell = 3$, at least two leaves are consecutive, say x_1 and x_2 . Since $cw(x_1wx_2) < \pi$ we add (x_1, x_2) to M and handle the rest like the case when $\ell = 1$.

$k = 3$ if $\ell = 2$ remove x_2 from T . Handle the rest as Case 3 in Step 2.

$k = 4$ if $\ell = 1$ remove x_1 from T . Handle the rest as Case 4 in Step 2.

$k = 5$ add (v_5, a_5) to M , remove a_5, b_5, v_5 from T , and handle the rest as Case 4 in Step 2.

This concludes the algorithm.

Lemma 16. *The convex empty regions that are considered in different iterations of the algorithm, do not overlap.*

Proof. In Step 1, Step 2 and the base cases, we used three types of convex empty regions; see Figure 14. Using contradiction, suppose that two convex regions P_1 and P_2 overlap. Since the regions are empty, no vertex of P_1 is in the interior of P_2 and vice versa. Then, one of the edges in $MST(P)$ that is shared with P_1 intersects some edge in $MST(P)$ that is shared with P_2 , which is a contradiction. ■

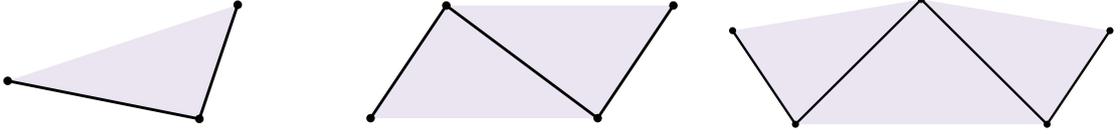


Figure 14: Empty convex regions. Bold edges belong to $MST(P)$.

Theorem 3. *Let P be a set of n points in the plane, where n is even, and let λ^* be the minimum bottleneck length of any plane perfect matching of P . In $O(n \log n)$ time, a plane matching of P of size at least $\frac{2n}{5}$ can be computed, whose bottleneck length is at most $(\sqrt{2} + \sqrt{3})\lambda^*$.*

Proof. Proof of planarity: In each iteration, in Step 1, Step 2, and in the base cases, the edges added to M are edges of $MST(P)$ or edges inside convex empty regions. By Lemma 16 the convex empty regions in each iteration will not be intersected by the convex empty regions in the next iterations. Therefore, the edges of M do not intersect each other and M is plane.

Proof of matching size: In Step 1, in each iteration, all the vertices which are excluded from T are matched. In Step 2, when $k = \ell = 1$ we match four vertices out of five, and when $k = 3, \ell = 0$ we match eight vertices out of ten. In base case (a) when $t = 5$ we match four vertices out of five. In base case (b) when $k = 3, \ell = 0$ we match eight vertices out of ten. In all other cases of Step 2 and the base cases, the ratio of matched vertices is more than $4/5$. Thus, in each iteration at least $4/5$ of the vertices removed from T are matched and hence $|M_i| \geq \frac{2n_i}{5}$. Therefore,

$$|M| = \sum_{i=1}^k |M_i| \geq \sum_{i=1}^k \frac{2n_i}{5} = \frac{2n}{5}.$$

Proof of edge length: By Lemma 11 the length of edges of T is at most λ^* . Consider an edge $e \in M$ and the path δ between its end points in T . If e is added in Step 1, then $|e| \leq 2\lambda^*$ because δ has at most two edges. If e is added in Step 2, δ has at most three edges ($|e| \leq 3\lambda^*$) except in Case 4. In this case we look at δ in more detail. We consider the worst case when all the edges of δ have maximum possible length λ^* and the angles between the edges are as big as possible; see Figure 15. Consider the edge $e = (a_1, b_2)$ added to M in Case 4. Since v_2 is an anchor and $cw(wv_2a_2) \geq \pi/3$, the angle $cw(b_2v_2w) \leq 2\pi/3$. As our choice between P_1 and P_3 in Case 4, $cw(w, v_1, a_1) \leq \pi/2$. Recall that e avoids w , and hence $|e| \leq (\sqrt{2} + \sqrt{3})\lambda^*$. The analysis for the base cases is similar.

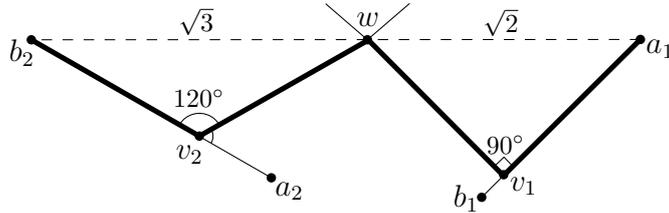


Figure 15: Path δ (in bold) with four edges of length λ^* between end points of edge (a_1, b_2) .

Proof of complexity: The Delaunay triangulation of P can be computed in $O(n \log n)$ time. Using Kruskal's algorithm, the forest F of even components can be computed in $O(n \log n)$ time. In Step 1 (resp. Step 2) in each iteration, we pick a leaf of T' (resp. T'') and according to the number of leaves (resp. anchors) connected to it we add some edges to M . Note that in each iteration we can update T, T' and T'' by only checking the two hop neighborhood of selected leaves. Since the two hop neighborhood is of constant size, we can update the trees in

$O(1)$ time in each iteration. Thus, the total running time of Step 1, Step 2, and the base cases is $O(n)$ and the total running time of the algorithm is $O(n \log n)$. ■

5 Conclusion

We considered the NP-hard problem of computing a bottleneck plane perfect matching of a point set. Abu-Affash et al. [1] presented a $2\sqrt{10}$ -approximation for this problem. We used the maximum plane matching problem in unit disk graphs (UDG) as a tool for approximating a bottleneck plane perfect matching. In Section 3.1 we presented an algorithm which computes a plane matching of size $\frac{n}{6}$ in UDG, but it is still open to show if this algorithm terminates in polynomial number of steps or not. We also presented a $\frac{2}{5}$ -approximation algorithm for computing a maximum matching in UDG. By extending this algorithm we showed how one can compute a bottleneck plane matching of size $\frac{n}{5}$ with edges of optimum-length. A modification of this algorithm gives us a plane matching of size at least $\frac{2n}{5}$ with edges of length at most $\sqrt{2} + \sqrt{3}$ times the optimum.

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