COMP 1805 Discrete Structures
Assignment 1

Due Tuesday, May 15th, 2012 at break (before 7:30pm)

Write down your name and student number on **every** page. The questions should be answered in order and your assignment sheets must be stapled, otherwise the assignment will not be marked. Total marks are 43.

1. (**4 marks**) For each of the following statements identify if the statement is a proposition, and if so what is its truth value?

   (a) $1 + 2 = 3$
   (b) Help me understand this logic stuff!
   (c) 7 is an even number.
   (d) $x + y = 4$

   Solution:

   (a) Yes, true.
   (b) No - not declarative.
   (c) Yes, false.
   (d) No - since it is neither true nor false.

   Grading:
   1 mark for each statement. 1/2 for propositions if truth value is not correctly indicated.

2. (**10 marks**) Prove or disprove the following: (you may use truth tables, or logical equivalences or any other valid form of argument).

   (a) $p \wedge q$ is equivalent to $\neg p \vee \neg q$
   (b) $\neg(p \vee \neg q)$ is equivalent to $\neg p \vee q$
   (c) $\neg(p \rightarrow \neg q)$ is equivalent to $p \wedge \neg q$
   (d) $p \wedge (q \vee r)$ is equivalent to $(p \wedge q) \vee (p \wedge r)$
   (e) $p \oplus q$ is equivalent to $\neg p \oplus \neg q$

   Solution:

   (a) This statement is false. A counterexample is $p = F$ and $q = T$. We have $p \wedge q$ as $F \wedge T$, which is false, and $\neg p \vee \neg q$ as $\neg F \vee \neg T$ (i.e., $T \vee F$), which is true. We found truth values where the statements are not equivalent, so the statement is false.
   It's also possible to see this by using a truth table:

p	q	$p \wedge q$	$\neg p \vee \neg q$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

The columns below the statements do not match, so the statements are not equivalent.

(A quick way to see that this statement is false is to remember De Morgan's law. $\neg p \vee \neg q$ is equivalent to $\neg(p \wedge q)$ by De Morgan's law, and this can never be equivalent to $p \wedge q$.)

(b) This statement is false. A counterexample is $p = T$ and $q = T$. We have $\neg(p \vee \neg q)$ as $\neg(T \vee \neg T)$ (i.e., $\neg(T \vee F)$), which is false, and $\neg p \vee q$ as $\neg T \vee T$, which is true. We found truth values where the statments are not equivalent, so the statement is false.

This could also be verified using a truth table:

p	q	$p \vee \neg q$	$\neg(p \vee \neg q)$	$\neg p \vee q$
T	T	T	F	T
T	F	T	F	F
F	T	F	T	T
F	F	T	F	T

The columns below the statements do not match, so the statements are not equivalent.

(c) This statement is false. A counterexample is $p = T$ and $q = T$. We have $\neg(p \to \neg q)$ as $\neg(T \to \neg T)$ (i.e., $\neg(T \to F)$), which is true, and $p \wedge \neg q$ as $T \wedge \neg T$ (i.e., $T \wedge F$), which is false. We found truth values where the statements are not equivalent, so the statement is false.

This could also be verified using a truth table:

p	q	$(p \to \neg q)$	$\neg(p \to \neg q)$	$p \wedge \neg q$
T	T	F	T	F
T	F	T	F	T
F	T	T	F	F
F	F	T	F	F

The columns below the statements do not match, so the statements are not equivalent.

(d) This statement is true. By applying the law of distributivity to $p \wedge (q \vee r)$, we get $(p \wedge q) \vee (p \wedge r)$ immediately. This could also be verified using a truth table:

p	q	r	$q \vee r$	$p \wedge q$	$p \wedge r$	$p \wedge (q \vee r)$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	F	T	T
T	F	T	T	F	T	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

The columns below the statements match, so the statements are equivalent.

(e) This statement is true. We'll use a truth table to prove this.

p	q	$p \oplus q$	$\neg p \oplus \neg q$
T	T	F	F
T	F	T	T
F	T	T	T
F	F	F	F

The columns below the statements match, so the statements are equivalent.

Grading:
2 marks for each. Deduct 1/2 mark for a minor errors in TT or equivalences. Deduct 1/2 mark for incorrect conclusion. Deduct full mark for significant error but mostly correct.

3. (**2 marks**) Determine whether or not the associative law holds for $\oplus$ (exclusive or).

Solution:
We use a truth table to demonstrate that the associative law applies for exclusive or.

a	b	c	$(a \oplus b) \oplus c$	$a \oplus (b \oplus c)$
T	T	T	T	T
T	T	F	F	F
T	F	T	F	F
T	F	F	T	T
F	T	T	F	F
F	T	F	T	T
F	F	T	T	T
F	F	F	F	F

Therefore, since the truth values in the columns for $(a \oplus b) \oplus c$ and $a \oplus (b \oplus c)$ are identical for all combinations of the input variables, the associative law applies for exclusive or.

Grading:
1 mark for correct answer. 1 mark for proper reasoning (eg. uses proper reasoning to argue that associative law holds for exclusive or).

4. (**6 marks**) Determine if the following are tautologies, contradictions or contingencies. You cannot use truth tables to justify your answers. Use either logical equivalences or some other means that does not use truth tables.

   (a) $((a \vee c) \wedge (b \vee c)) \vee ((c \rightarrow \neg b) \wedge (c \rightarrow a))$

   (b) $\neg((\neg a \rightarrow (\neg b \vee c)) \leftrightarrow (b \rightarrow (a \vee c)))$

   (c) $((a \vee b) \wedge (a \rightarrow c)) \rightarrow (b \vee c)$

   (a) The statement is a tautology.

$$\begin{aligned}
& ((a \vee c) \wedge (b \vee c)) \vee ((c \rightarrow \neg b) \wedge (c \rightarrow a)) & \\
\equiv\ & ((a \wedge b) \vee c) \vee ((\neg c \vee \neg b) \wedge (\neg c \vee a)) & \text{Distributive and IE(x2).} \\
\equiv\ & ((a \wedge b) \vee c) \vee ((\neg c \vee (\neg b \wedge a)) & \text{Distributive.} \\
\equiv\ & (a \wedge b) \vee c \vee \neg c \vee (\neg b \wedge a)) & \text{Associative.} \\
\equiv\ & (a \wedge b) \vee T \vee (\neg b \wedge a) & \text{Negation law.} \\
\equiv\ & T & \text{Domination.}
\end{aligned}$$

   (b) The statement is a contradiction.

$$\begin{aligned}
& \neg((\neg a \rightarrow (\neg b \vee c)) \leftrightarrow (b \rightarrow (a \vee c))) & \\
\equiv\ & \neg((a \vee (\neg b \vee c)) \leftrightarrow (\neg b \vee (a \vee c))) & \text{Implication equivalence.} \\
\equiv\ & \neg((a \vee \neg b \vee c) \leftrightarrow (a \vee \neg b \vee c)) & \text{Commutative and associative laws.}
\end{aligned}$$

   This is always false. The compound proposition $(a \vee \neg b \vee c) \leftrightarrow (a \vee \neg b \vee c)$ is obviously always true since the two sides of the biconditional are equivalent. But we take the negation of this so the final result is a contradiction (always false).

   (c) The statement is a tautology.

$$\begin{aligned}
& ((a \vee b) \wedge (a \rightarrow c)) \rightarrow (b \vee c) & \\
\equiv\ & \neg((a \vee b) \wedge (\neg a \vee c)) \vee (b \vee c) & \text{Implication equivalence(x2).} \\
\equiv\ & (\neg(a \vee b) \vee \neg(\neg a \vee c)) \vee (b \vee c) & \text{De Morgans.} \\
\equiv\ & ((\neg a \wedge \neg b) \vee (\neg\neg a \wedge \neg c)) \vee (b \vee c) & \text{De Morgans.} \\
\equiv\ & ((\neg a \wedge \neg b) \vee (a \wedge \neg c)) \vee (b \vee c) & \text{Double negation.} \\
\equiv\ & (\neg a \wedge \neg b) \vee b \vee (a \wedge \neg c) \vee c & \text{Assocative and commutative.} \\
\equiv\ & ((\neg a \vee b) \wedge (\neg b \vee b)) \vee ((a \vee c)) \wedge (\neg c \vee c)) & \text{Distributive.} \\
\equiv\ & ((\neg a \vee b) \wedge T) \vee ((a \vee c)) \wedge T) & \text{Negation.} \\
\equiv\ & (\neg a \vee b) \vee (a \vee c) & \text{Identity laws (x2).} \\
\equiv\ & a \vee \neg a \vee b \vee c) & \text{Associative and commutative.} \\
\equiv\ & T \vee b \vee c & \text{Negation} \\
\equiv\ & T & \text{Domination}
\end{aligned}$$

Grading:
1/2 mark for correct answer, up to 1 1/2 marks for correct logic leading to correct (or incorrect) answer.

5. (**2 marks**) Express the statement $a \oplus b$ using the other logical operators (eg. $\neg$, $\vee$, $\wedge$, etc.). In other words, write a logical statement that is equivalent to $a \oplus b$ without using $\oplus$.

   Solution:
   The statement $a \oplus b$ is equivalent to $(a \vee b) \wedge \neg(a \wedge b)$. Alternately applying De Morgans to $(a \vee b) \wedge \neg(a \wedge b)$ gives us $(a \vee b) \wedge (\neg a \vee \neg b)$. Either of these statements, or something equivalent is correct.

   Grading:
   1 mark for good effort! 2 marks for correct effort (one of the above or something logically equivalent.

6. (**4 marks**) Translate the following statements into English where $W(x)$ is "x has wings", $F(x)$ is "x can fly", and $Q(x)$ is "x quacks", and the domain is all animals.

   (a) $\exists x(W(x) \wedge \neg F(x))$.
   (b) $\forall x\,[(W(x) \wedge Q(x)) \rightarrow F(x)]$.

   (a) There exists an animal with wings that cannot fly.
   (b) All animals that have wings and quack can fly.

   Grading:
   Full two marks for correct sentence of equivalent, drop part marks if student is close (eg. get and and/or mixed up but otherwise correct).

7. (**5 marks**) Translate the following English statements into propositional logic. Define the propositions you will use. Your base propositions should not be negative or compound propositions themselves. Use brackets when necessary to make the order of evaluation clear.

   (a) If I study Discrete Math and Linear Algebra then I will get good grades.
   (b) The sky is blue or the earth is round.
   (c) It is Monday and I am not at work.
   (d) I will live to be 100 years old iff I excercise every day and I eat well.
   (e) I miss the bus if I sleep in.

   Solution:

   (a) p: I study Discrete Math
   q: I study Linear Algebra
   r: I will get good grades
   Statement is $(p \wedge q) \rightarrow r$.
   (b) b: The sky is blue.
   r: The earth is round.
   Statement is $b \vee r$.
   (c) m: It is Monday.
   w: I am at work.
   Statement is $m \wedge \neg w$.
   (d) p: I will live to be 100 years old.
   q: I exercise every day.
   r: I eat well.
   Statement is $p \leftrightarrow (q \wedge r)$.
   (e) p: I miss the bus.
   q: I sleep in.
   Statement is $q \rightarrow p$.

   Grading:
   1 mark for each correct, 1/2 mark if 'close'.

8. (**2 marks**) Give the negation of the following statement. The negation symbols should proceed the predicates.
$\exists x \forall y \exists z\, [A(x,y) \rightarrow (B(x,y) \vee C(x,y))]$
Solution:

$$\begin{array}{lll} & \neg \exists x \forall y \exists z\, [A(x,y) \rightarrow (B(x,y) \vee C(x,y))] & \\ \equiv & \forall x \exists y \forall z \neg\, [A(x,y) \rightarrow (B(x,y) \vee C(x,y))] & \\ \equiv & \forall x \exists y \forall z \neg\, [\neg A(x,y) \vee B(x,y) \vee C(x,y)] & \text{Implication Equivalence} \\ \equiv & \forall x \exists y \forall z\, [A(x,y) \wedge \neg B(x,y) \wedge \neg C(x,y)] & \text{De Morgans} \end{array}$$

Grading:
2 marks for correct answer. 1 mark if they apply negation properly but don't have negation signs preceeding predicates (ie. they only do the first step).

9. (**6 marks**) Let $H(x)$ be "x plays hockey", $S(x)$ be "x can skate", $D(x)$ be "x can dance" and $E(x)$ be "x earns money". The universe of discourse is all humans. State the following logically.

   (a) Everyone that can skate and dance earns money.
   (b) At most two people can play hockey and dance.
   (c) Not everyone who plays hockey earns money.

   Solution:

   (a) $\forall x(S(x) \wedge D(x) \rightarrow E(x))$
   (b) $\neg(\exists x \exists y \exists z((H(x) \wedge D(x)) \wedge (H(y) \wedge D(y)) \wedge (H(z) \wedge D(z)) \wedge (x \neq y) \wedge (y \neq z) \wedge (x \neq z))$. By applying DeMorgan's Law, we see that this is equivalent to all the following statements:
      i. $\forall x \neg(\exists y \exists z((H(x) \wedge D(x)) \wedge (H(y) \wedge D(y)) \wedge (H(z) \wedge D(z)) \wedge (x \neq y) \wedge (y \neq z) \wedge (x \neq z)))$.
      ii. $\forall x \forall y \neg(\exists z((H(x) \wedge D(x)) \wedge (H(y) \wedge D(y)) \wedge (H(z) \wedge D(z)) \wedge (x \neq y) \wedge (y \neq z) \wedge (x \neq z)))$.
      iii. $\forall x \forall y \forall z(\neg(H(x) \vee \neg D(x)) \vee (\neg H(y) \vee \neg D(y)) \vee (\neg H(z) \vee \neg D(z)) \vee (x = y) \vee (y = z) \vee (x = z)))$.
   (c) $\neg(\forall x(H(x) \rightarrow E(x)))$. By DeMorgans, we see that this is also equivalent to $\exists x(H(x) \wedge \neg E(x))$

   Grading:
   2 marks for a correct solution (part marks if they are close).

10. (**2 marks**) Let $L(x)$ be "x is a Lion" and $M(x)$ be "x eats meat.", where the universe of discourse is all animals. Are the statements "$\forall x(L(x) \rightarrow M(x))$" and "$\forall x(L(x) \wedge M(x))$" logically the same thing? Prove your answer.
    Solution:
    The question asks if the statement "All lions eat meat" is equivalent to "all animals are lions who eat meat." Clearly these are different since all animals are not necessarily lions. We now prove this intuition logically.

    To logically prove that the two statements are not equivalent, we need to construct a counter-example. Let a be an animal such that $L(a) = F$ and $M(a) = T$.

    For the first statement, we have
    $$L(a) \rightarrow M(a) \equiv F \rightarrow T \equiv T. \tag{1}$$

    However, for the second statement, we have
    $$L(a) \wedge M(a) \equiv F \wedge T \equiv F. \tag{2}$$

    Therefore, since $L(a) \rightarrow M(a) \not\equiv L(a) \wedge M(a)$, the two statements are not equivalent.

    Grading:
    Give 1 1/2 marks if the student correctly states that the statements are not equivalent and argues this correctly. Getting a valid proof gives the final 1/2 mark.