COMP 1805 Discrete Structures
Assignment 2

Due Thursday, May 24th, 2012 at break

Write down your name and student number on **every** page. The questions should be answered in order and your assignment sheets must be stapled, otherwise the assignment will not be marked. Total marks are 37.

1. (**3 marks**) Use rules of inference to determine if ”I will feel tired” is a valid conclusion form the following premises ”If it is the weekend I will go shopping”, ”It is the weekend or I will go out to dinner”, ”If I go out to dinner then I will eat too much”, ”If I eat too much I will feel tired”, and ”I will not go shopping”.

   Solution:
   First we determine our propositions:

   - w: It is the weekend.
   - s: I will go shopping
   - d: I will go out for dinner.
   - e: I will eat too much.
   - t: I will feel tired.

   Now, we apply rules of inference.

Step	Reason
1. $\neg s$	Hypothesis
2. $w \rightarrow s$	Hypothesis
3. $\neg w$	Modus tollens from 1 and 2
4. $w \vee d$	Hypothesis
5. d	Disjunctive syllogism from 4 and 3
6. $d \rightarrow e$	Hypothesis
7. e	Modus ponens form 5 and 6
8. $e \rightarrow t$	Hypothesis
9. t	Modus ponens from 7 and 8

   Grading:
   1 Mark for getting list of propositions correct. 2 For applying rules of inference correctly.

2. (**3 marks**) Use rules of inference for quantifiers to show that the statement ”Andrew can count by twos and tie his shoes” is true for Andrew the university student. The premises are ”All university students that have finished high school can count to fifty and tie their shoes”, ”All university students have finished highschool and can count by twos”.

   Solution:
   First we list our propositions.

   - Let H(x) denote ”x has finished highschool”.
   - Let F(x) denote ”x can count to fifty”.
   - Let S(x) denote ”x can tie thier shoes”.

- Let T(x) denote ”x can count by twos”.

In the arguments that follow *a* is used to represent an arbitrary element of the domain. This could also be argued using Andrew instead of an arbitrary student.

Step	Reason
1. $\forall x\,[H(x) \to (F(x) \wedge S(x))]$	Premise
2. $\forall x(H(x) \wedge T(x))$	Premise
3. $H(a) \to (F(a) \wedge S(a))$	Universal instantiation from 1
4. $H(a) \wedge T(a)$	Universal instantiation from 2.
5. $H(a)$	Simplification from 4.
6. $F(a) \wedge S(a)$	Modus ponens from 5 and 3.
7. $S(a)$	Simplification from 6.
8. $T(a)$	Simplification from 4.
9. $S(a) \wedge T(a)$	Conjuction from 7 and 8
10. $\forall x(S(x) \wedge T(x))$	Universal generalization from 9
11. $S(Andrew) \wedge T(Andrew)$	Universal instantiation from 10.

Grading:
1 Mark for getting list of propositions correct. 2 For applying rules of inference correctly.

3. (**3 marks**) Prove or disprove that $1 + 3\sqrt{2}$ is irrational.

   Solution:
   We give a proof by contradiction. Assume that $1 + 3\sqrt{2}$ is rational then (assume $b \neq 0$):

$$\begin{aligned} 1 + 3\sqrt{2} &= \frac{a}{b} \\ 3\sqrt{2} &= \frac{a}{b} - 1 \\ 3\sqrt{2} &= \frac{a}{b} - \frac{b}{b} \\ 3\sqrt{2} &= \frac{a-b}{b} \\ \sqrt{2} &= \frac{a-b}{3b} \end{aligned}$$

   However, $a - b$ and $3b \neq 0$ are integers, which means $\sqrt{2}$ must be rational, but we know that $\sqrt{2}$ is irrational, so we have contradicted out claim.

   Grading:
   Full marks for correct proof, part marks for progress towards correct answer.

4. (**2 marks**) Prove or disprove the following: If x and y are rational, then x^y is rational.

   Solution:
   The claim is false. One possible counterexample is the following. Let $x = 2 = \frac{2}{1}$, which is rational. Let $y = \frac{1}{2}$, which is rational. Now, $x^y = 2^{\frac{1}{2}} = \sqrt{2}$, which is irrational.

   Grading:
   Give 1/2 mark for correct conclusion. 1 1/2 marks for the correct proof (or some other correct proof), again give up to 1 mark for a good effort that demonstrates understanding of what the student wants to prove, but doesn’t quite work.

5. (**2 marks**) Prove or disprove that if set A and set B both have the same cardinality, then the power set of A is identical to the power set of B.

   Solution:
   False. Let $A = \{1, 2\}$ and $B = \{a, b\}$. Both A and B have the same cardinality. However, the powerset of A is $\{\{\}, \{1\}, \{2\}, \{1, 2\}\}$ and the powerset of B is $\{\{\}, \{a\}, \{b\}, \{a, b\}\}$ which are different.

Grading:
Give 1/2 mark for correct conclusion. 1 1/2 marks for the correct proof (or some other correct proof), again give up to 1 mark for a good effort that demonstrates understanding of what the student wants to prove, but doesn't quite work.

6. (**6 marks**) Determine whether or not the following are valid. Justify your answer by using set identities or membership tables as indicated. Let A, B and C be sets.

   (a) $(A \oplus B) = (A - B) \cup (B - A)$ (membership table).

   (b) $(A - B) \cup (B - A) = \overline{(A \cap B)}$ (membership table).

   (c) $(C - A) \cap (B - A) = (C \cap B) \cap \overline{A}$ (set identities).

Solution:

(a)

A	B	$A-B$	$B-A$	$(A-B)\cup(B-A)$	$(A\oplus B)$	$(A\oplus B)=(A-B)\cup(B-A)$
1	1	0	0	0	0	1
1	0	1	0	1	1	1
0	1	0	1	1	1	1
0	0	0	0	0	0	1

Since the last column is all filled with ones, the identity is valid.

(b)

A	B	$A-B$	$B-A$	$(A-B)\cup(B-A)$	$A\cap B$	$\overline{(A\cap B)}$	$(A-B)\cup(B-A)=\overline{(A\cap B)}$
1	1	0	0	0	1	0	1
1	0	1	0	1	0	1	1
0	1	0	1	1	0	1	1
0	0	0	0	0	0	1	0

Since the last column contains a zero, the identity is not valid.

(c)

$$
\begin{aligned}
(C-A)\cap(B-A) &= (C\cap\overline{A})\cap(B\cap\overline{A}) && \text{(Definition of set difference)}\\
&= C\cap(\overline{A}\cap B)\cap\overline{A} && \text{(Associative laws)}\\
&= C\cap(B\cap\overline{A})\cap\overline{A} && \text{(Commutative laws)}\\
&= (C\cap B)\cap(\overline{A}\cap\overline{A}) && \text{(Associative laws)}\\
&= (C\cap B)\cap\overline{A} && \text{(Idempotent laws)}
\end{aligned}
$$

Grading:
2 Marks for each question. Part marks if minor mistakes lead to incorrect answer.

7. (**6 marks**) Draw the Venn Diagram of the following:

   (a) $(A \oplus B)$

   (b) $\overline{(B \cap C)} \cup \overline{A}$

   (c) $(B \cap C) \cup \overline{A \cap B}$

Solution:

(a) $(A \oplus B)$ see Figure below.

(b) $\overline{(B \cap C)} \cup \overline{A}$ see Figure below.

(c) $(B \cap C) \cup \overline{A \cap B}$ see Figure below.

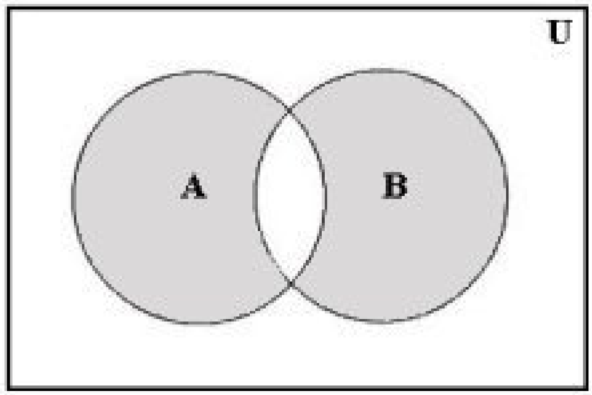

Figure 1: Solution to 7(a)

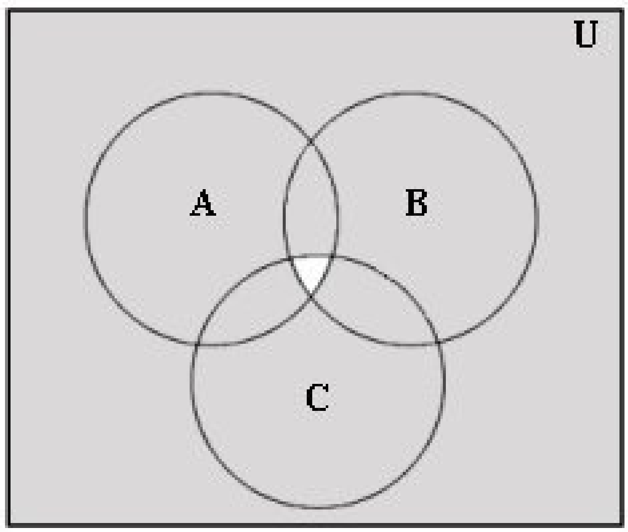

Figure 2: Solution to 7(b)

Grading:
2 marks each. Again part marks for 'close'.

8. (**2 marks**) Prove or disprove that if set A and set B both have the same power set, then $A = B$.

Solution:
We will prove this by contradiction. Assume for a contradiction that $A \neq B$. This means that either A contains an element x that is not in B or A is a proper subset of B. If A contains an element x that is not in B, then $\{x\} \in P(A)$ but $\{x\} \notin P(B)$ which contradicts the fact that $P(A) = P(B)$. If A is a proper subset of B, then $|A| < |B|$. But then $|P(A)| = 2^{|A|} < 2^{|B|} = |P(B)|$. Again, we contradict the fact that $P(A) = P(B)$. Therefore, we conclude that $A = B$ as required.

Grading:
2 marks for correct, part marks for reasonable efforts.

9. (**8 marks**) Indicate if each of the following statements is true or false. As was done in class you should provide witnesses to demonstrate when a function is of the order specified.

   (a) $3n^4 + 107n^3 + 5$ is $O(n^5)$

   (b) $17n^2 + 23n + 2$ is $O(n^4)$

   (c) $\frac{1}{4}n \log n$ is $O(n)$

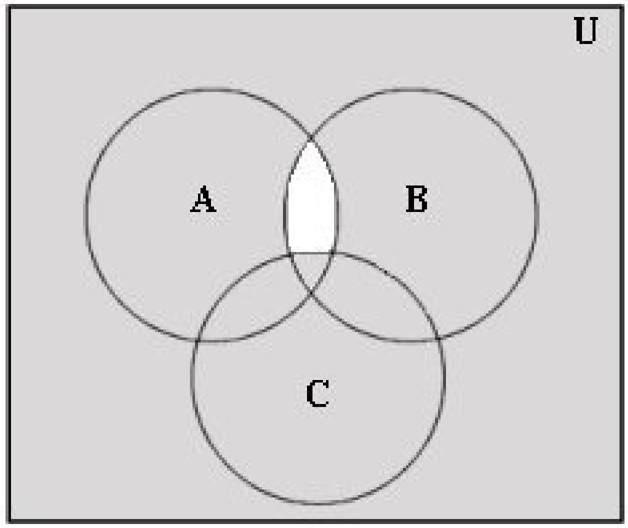

Figure 3: Solution to 7(c)

(d) $45 + 1/n$ is $\Theta(1)$

Solution:

(a) We need to show that $3n^4 + 107n^3 + 5 \le cn^5,\ \forall n \ge d$ where we choose the values $c > 0$ and d.

$$\begin{aligned} 3n^4 + 107n^3 + 5 &\le 3n^5 + 107n^3 + 5 \text{ when } n \ge 1 \\ &\le 3n^5 + 107n^5 + 5n^5 \text{ when } n \ge 1 \\ &\le 115n^5\ \forall n \ge 1 \end{aligned}$$

(b) We need to show that $17n^2 + 23n + 2 \le cn^4,\ \forall n \ge d$ where we choose the values $c > 0$ and d.

$$\begin{aligned} 17n^2 + 23n + 2 &\le 17n^2 + 2 \text{ when } n \ge 1 \\ &\le 17n^2 + 23n^2 + 2n^2 \text{ when } n \ge 1 \\ &= 42n^2 \text{ when } n \ge 1 \\ &\le 42n^4\ \forall n \ge 1 \end{aligned}$$

(c) So we need to show:

$$\begin{aligned} \frac{1}{4} n \log_2 n &\le c_1 n \text{ when } n \ge d \\ \frac{1}{4} \log_2 n &\le c_1 \text{ divide both sides by } n \\ \log_2 n &\le 4 \cdot c_1 \text{ divide both sides by } n\text{, then multiply by 4} \\ \log_2 n &\le c_2 \text{ where } c_2 = 4 \cdot c_1 \end{aligned}$$

Notice that $\log_2 n$ can not be smaller than a fixed number c_2 for infinitely large n. Therefore, this is impossible.

(d) We must show that $45 + 1/n$ is $O(1)$ and $\Omega(1)$.

To show that $45 + 1/n$ is $O(1)$ we must find constants c and d such that $45 + 1/n \leq c,\ \forall n \geq d$. Thus we can select thes values as follows.

$$\begin{aligned} 45 + 1/n &\leq 45 + 1 \text{ when } n \geq 1 \\ &\leq 46\ \forall n \geq 1 \end{aligned}$$

Therefore, this is true with $c = 46$ and $d = 1$ (any $c46$ will do).

To show that $45 + 1/n$ is $\Omega(1)$ we must find constants c and d such that $45 + 1/n \geq c,\ \forall n \geq d$. We can select and $c \leq 45$ and since $1/n$ is always greater than $0\ \forall n \geq 1$, we have $45 + 1/n$ is $\Omega(1)$.

Grading:
Drop 1/2 mark if the student gets a value for c but omits d. Give part marks for reasonable efforts that show student is working towards the correct answer even if they make a minor mistake.

10. (**2 marks**) If $f(x)$ is $\Omega(g(x))$ and $f(x)$ is $O(h(x))$ then is it true that $g(x)$ is $O(h(x))$?

Solution:
Yes. Since $C_1|g(x)| \leq |f(x)|$ due to $f(x)$ is $\Omega(g(x))$. Then we have that $|f(x)| \leq C_2|h(x)|$ so $C_1|g(x)| \leq |f(x)| \leq C_2|h(x)|$ so $|g(x)| \leq \frac{C_2}{C_1}|h(x)|$ and $g(x)$ is $O(h(x))$.

Grading:
Give part marks for an incorrect answer that shows some evidence the student understands what they are trying to prove.