COMP 1805 Discrete Structures
Assignment 3

Due Thursday, June 7th 2012 at the break

Write down your name and student number on **every** page. The questions should be answered in order and your assignment sheets must be stapled, otherwise the assignment will not be marked. Total marks are 51.

1. (**4 marks**) Compute the exact sums of the following summations. You must show your work so you will not be awarded any marks for simply writing down a number for your answer - regardless of whether it is right or wrong.

   (a) $\sum_{k=1}^{n}(8k + 4 + 12k^2)$

   (b) $\sum_{j=13}^{29} 6j$

   (c) $\sum_{i=1}^{6} 4 \cdot 2^{i-1}$

   (d) $\sum_{x=0}^{10}\sum_{y=0}^{3}(xy + x + y + 1)$

   Solution:

   (a)

$$\begin{aligned}
\sum_{k=1}^{n} 8k + 4 + 12k^2 &= 8\sum_{k=1}^{n} k + 4n + 12\sum_{k=1}^{n} k^2 \\
&= 8 \cdot \left(\frac{n(n+1)}{2}\right) + 12 \cdot \left(\frac{n(n+1)(2n+1)}{6}\right) + +4n \\
&= 4n(n+1) + 2n(n+1)(2n+1) + 4n \\
&= (4n^2 + 4n) + (4n^3 + 6n^2 + 2n) + 4n \\
&= 4n^3 + 10n^2 + 10n
\end{aligned}$$

   (b)

$$\begin{aligned}
\sum_{j=13}^{29} 6 \cdot j &= 6\sum_{j=13}^{29} j \\
&= 6\left(\sum_{j=1}^{29} j - \sum_{j=1}^{12} j\right) \\
&= 6\left(\frac{29(29+1)}{2} - \frac{12(12+1)}{2}\right) \\
&= 6(435 - 78) \\
&= 2142
\end{aligned}$$

(c) If we let $k = i - 1$ then we have:

$$\sum_{k=0}^{5} 4 \cdot 2^k = \sum_{i=1}^{6} 4 \cdot 2^{i-1}$$

This is a the sum of a geometric progression with $a = 4$ and $r = 2$, so we have:

$$\sum_{k=0}^{5} 4 \cdot 2^k = \frac{4 \cdot 2^6 - 4}{2 - 1} = \frac{256 - 4}{1} = 252$$

(d)

$$\begin{aligned}
\sum_{x=0}^{10}\sum_{y=0}^{3}(xy + x + y + 1) &= \sum_{x=0}^{10}\sum_{y=0}^{3}(x+1)(y+1) \\
&= \sum_{x=0}^{10}\sum_{k=1}^{4}(x+1)k \\
&= \sum_{x=0}^{10}((x+1) + 2(x+1) + 3(x+1) + 4(x+1)) \\
&= \sum_{x=0}^{10}(10x + 10) \\
&= \sum_{x=0}^{10} 10x + \sum_{x=0}^{10} 10 \\
&= \left[10\sum_{x=0}^{10} x\right] + 110 \\
&= \left[10\sum_{x=1}^{10} x\right] + 110 \\
&= \left[10 \cdot \frac{10(10+1)}{2}\right] + 110 \\
&= 550 + 110 \\
&= 660
\end{aligned}$$

Grading:

2. (**2 marks**) A wheel of fortune has the integers from 1 to 25 placed on it in a random fashion. Show that no matter how the numbers are arranged on the wheel there will always be at least one grouping of three consecutive numbers which sum to at least 39.

Solution:
We can show this is true using summations and a proof by contradiction. First we sum the total of all numbers on the wheel as follows:

$$\sum_{k=1}^{25} = \frac{25 \cdot (25+1)}{2} = 325 \tag{1}$$

Now consider some arbitrary ordering of the numbers 1 through 25 on the wheel, we will assume that no 3 consecutive numbers have a sum of greater than or equal to 39. Next find the number 1 on the wheel, and starting with the next number in a clockwise direction[1] group the remaining numbers on the wheel into groups

[1] direction doesn't really matter, just have to pick one

of three consecutive numbers. There will be $24/3 = 8$ such groups. We have claimed that no such group has a sum 39 or more, so at most a group can have a sum of 38. So there are 8 groups of at most 38, which can sum to at most $38 \cdot 8 = 304$. Adding the number 1 to this value we have a total of 305 which is less than the sum of 325. So all the groups cannot sum to less than 39 which is a contradiction of our previous claim. ∎

Grading:

3. (**4 marks**) Use mathematical induction to show that $\sum_{j=0}^{n}(j+1) = \frac{(n+1)(n+2)}{2}$.

Solution:

BASIS STEP: For $j = 0$ we have $\sum_{j=0}^{n}(j+1) = 0 + 1 = 1$ and:

$$\frac{(n+1)(n+2)}{2} = \frac{(1)(2)}{2} = 1$$

INDUCTIVE HYP: Assume $\sum_{j=0}^{k}(j+1) = \frac{(k+1)(k+2)}{2}$ for some $k > 0$.

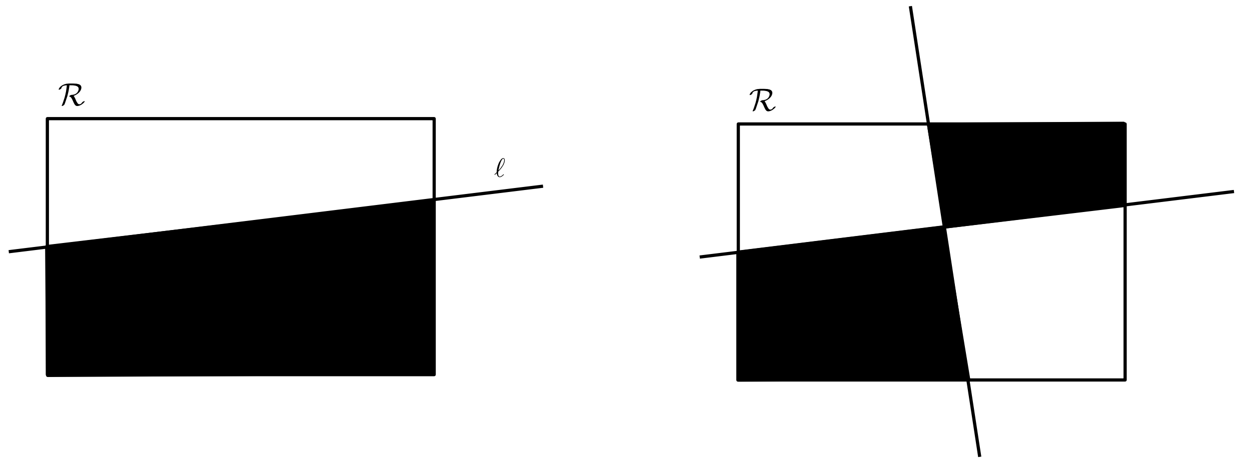
INDUCTIVE STEP: For the inductive step we wish to show that if the inductive hypothesis holds then $\sum_{j=0}^{k+1}(j+1) = \frac{([k+1]+1)([k+1]+2)}{2}$ also holds. Which we can do as follows:

$$\begin{aligned}
\sum_{j=0}^{k+1}(j+1) &= \sum_{j=0}^{k}(j+1) + [k+1] + 1 \\
&= \frac{(k+1)(k+2)}{2} + (k+2) \text{By inductive hypothesis} \\
&= \frac{(k+1)(k+2)}{2} + \frac{2(k+2)}{2} \\
&= \frac{(k+1)(k+2) + 2(k+2)}{2} \\
&= (k+2)\left[\frac{(k+1)+2}{2}\right] \\
&= (k+2)\left[\frac{(k+3}{2}\right] \\
&= \frac{(k+2)(k+3)}{2} \\
&= \frac{([k+1]+1)([k+1]+2)}{2}
\end{aligned}$$

∎

Grading:

4. (**4 marks**) Assume we have a rectangle, $\mathcal{R}$ through which we can draw any number of lines. The first line we draw (we assume any line we draw intersects $\mathcal{R}$) subdivides $\mathcal{R}$ into two *faces*. Any subsequent line we draw will likewise split one or more of the existing faces, and create a new face(s). Any face we intersect is split into exactly two new faces. We say the lines meet at a *vertex* and that portion of a line between two *vertices* is an *edge*. Two faces are *adjacent* if and only if they share an edge. Now assume we want to colour the faces in $\mathcal{R}$ black or white. Prove using induction that it is always possible to colour the faces in $\mathcal{R}$ such that no two adjacent faces are coloured the same. Figure 4 below shows such a subdivision and colouring with one and two lines.



(a) (b)

Solution:
BASIS STEP: For $P(1)$ obviously we split $\mathcal{R}$ into two faces, we colour one white and the other black.

INDUCTIVE HYP: Assume the for $P(k)$ lines we can always colour the faces black and white without any adjacent faces having the same colour.

INDUCTIVE STEP: We now show when adding an extra line such that we have $P(k+1)$ lines we can always maintain the property that no two adjacent faces have the same colour. Consider what happens when we draw the (k+1)st line, which we can call ℓ. It intersects one or more of the faces in $\mathcal{R}$, and in fact divides the faces of $\mathcal{R}$ into two sets of faces, those to its right and those to its left (if it is horizontal we can use above and below). Any face that doesn't touch ℓ will have a valid colour, but if we consider the set of faces intersected by ℓ we need to split those in two and then re-colour one of the new faces (since the two new faces are clearly adjacent). But we have a problem. Say we have intersected a white face, we now have two faces, both of which are adjacent to only black faces, but the new faces are adjacent so we cannot colour them both white. The following technique however solves the problem. We draw ℓ then we re-colour ALL faces on its left with the opposite colour (if its black make it white, and vice versa). The new faces are OK because the colour of the left half changes and the right stays the same. Furthermore, since we change all faces on the left side no adjacent faces will have the same colour, and on the right the colouring didn't change so there is no problem there either. ∎

Grading:
1 mark for setting things up properly, basis step, inductive hypothesis, and inductive step. Correct basis step Correct inductive hypthesis Correct inductive step.

5. (**4 marks**) Use mathematical induction to prove that $n! \leq n^n$.

Solution:
BASIS STEP: When $n = 1$ we have $1! \leq 1^1$ so the claim is true for $n = 1$.

INDUCTIVE HYP: $k! \leq k^k$.

INDUCTIVE STEP: Assume that $k! \leq k^k$ is true for an arbitrary k. We will show that $(k+1)! \leq (k+1)^{k+1}$.

$$\begin{array}{rcll} (k+1)^{k+1} & = & (k+1)\cdot(k+1)\cdot\ldots\cdot(k+1) & k+1 \text{ times} \\ & > & k\cdot k\cdot k\cdot\ldots\cdot k\cdot(k+1) & \text{with } k\ k \text{ terms.} \\ & = & k^k\cdot(k+1) & \\ & > & k!\cdot(k+1) & \text{since } k! \leq k^k \\ & = & (k+1)! & \end{array}$$

So $(k+1)^{k+1} > (k+1)!$ which completes our proof. ∎

Grading:

6. (**4 marks**) Assume that you live in a country where the only currency are 3\$ and \$10 dollar bills. Show that every amount greater than \$17 can be made from these bills.

Let $P(n)$ be the statement that a postage of n cents can be formed using just 4 and 7 cent stamps. Use *strong induction* to prove that $P(n)$ is true for $n \geq 18$.

Solution: First question.
BASIS STEP: $P(18) = 3 \times 6$ and $P(19) = 10 + 3 \times 3$ and $P(20) = 10 \times 2$ and $P(21) = 3 \times 7$.

INDUCTIVE HYP: Assume $P(j)$ is true for $18 \leq j \leq k$ where $k \geq 21$.

INDUCTIVE STEP: We want to show we can make currency of $P(k+1)$ dollars. From the inductive hypothesis we know that we can make change for $P(k-2)$ because $k - 2 \geq 18$. Thus we can make change for $P(k-2)$ using just \$3 and \$10 bills. So we take currency for $P(k-2)$ and add a \$3 bill and we have currency for for $P(k+1)$ dollars. ∎

Solution: Second question.
BASIS STEP: $P(18) = 7 + 7 + 4$ and $P(19) = 4 + 4 + 4 + 7$ and $P(20) = 4 \times 5$ and $P(21) = 3 \times 7$.

INDUCTIVE HYP: Assume $P(j)$ is true for $18 \leq j \leq k$ where $k \geq 21$.

INDUCTIVE STEP: We want to show we can make postage for $P(k+1)$ cents. From the inductive hypothesis we know that we can make postage for $P(k-3)$ cents because $k - 3 \geq 18$. Thus we can make postage for $P(k-3)$ using just 4 and 7 cent stamps. So we take stamps for $P(k-3)$ and add a 4 cent stamp and we have postage for for $P(k+1)$ cents. ∎

Grading:
There was only supposed to be one question here, so give full marks if they get one (or both) correct.

7. (**3 marks**) Provide a recursive definition for the following sequences where a_n, $n = 1, 2, 3, \ldots$:

   (a) $a_n = 4$
   (b) $a_n = \frac{n}{2} + 3$
   (c) $a_n = n!$

   Solution:

   (a) $a_0 = 4$, $a_n = a_{n-1}$
   (b) $a_0 = 3$, $a_n = a_{n-1} + \frac{1}{2}$
   (c) $a_0 = 1$, $a_n = n \cdot a_{n-1}$

   Grading:
   1 mark each for correct answer, part marks if students is 'close'.

8. (**2 marks**) How many bits strings are there of length less than ten and with an odd number of bits?

   Solution:
   Each bit has 2 possible values, 0 or 1. The bit strings of length less than ten with an odd number of bits are of lengths $1, 3, 5, 7, 9$. There are 2^1 bit strings of length 1, 2^3 of length 3, 2^5 of length 5, 2^7 of length 7, and 2^9 of length 9 so the total number of bit strings is $2^1 + 2^3 + 2^5 + 2^7 + 2^9$.

   Grading: Give full 2 marks for correct answer. 1 mark for partial answer (eg. figure out bit stings of only one of the lengths).

9. (**2 marks**) How many numbers between 0 (000,000 use 0 as a place holder for numbers less than 100,000) and 999,999 contain at least two even digits. Assume that placeholder 0's count as even digits (eg 111,123 has one even digit and 001,234 has 4 even digits).

   Solution:
   The easiest way to solve this is to use the complement method. There are 10^6 such numbers in total. We will calculate the how many numbers have zero or just 1 even number and remove that from the total.

   There are 5^6 numbers made up of just odd digits. There are $\binom{6}{1}$ ways to pick a position for an even digit times 5 possible even digits, times 5^5 ways to place the other 5 odd digits. So there are $\binom{6}{1} \cdot 5^1 \cdot 5^5$ numbers with just a single even digit. So in total the number of digits with at least two even digits is $10^6 - 5^6 - \binom{6}{1} \cdot 5^1 \cdot 5^5$.

   Grading: Again 2 marks for correct answer, part marks for partially correct answer.

10. (**2 marks**) How many integers greater than 100 and less than 1000 are divisibile by 3, 4, or 7?

Solution:
Get out your calculators!

This is an inclusion-exclusion problem. We first need to figure out how many values > 100 and < 1000 are divisible by 3, 4, and 7 respectively. But then to avoid double counting we must remove those numbers divisible by 3 and 4, by 3 and 7, and by 4 and 7. The integers divisible by 3 and 4 are the same as the numbers divisible by $3 \times 4 = 12$, and likewise those divisible by 3 and 7 are divisible by 21, while those divisible by 4 and 7 are divisible by 28. Finally, after removing the numbers divisible by 12, 21 and 28 we must add back in the numbers divisible by 3 and 4 and 7, which are divisible by $3 \times 4 \times 7 = 84$. In summary if D_j denotes the number of integers greater than 100 and less than 1000 divisible by j then we have our total as:

$$D_{3 \vee 4 \vee 7} = D_3 + D_4 + D_7 - (D_{12} + D_{21} + D_{28}) + D_{84}$$

So what is D_3? Well we can calculate D_3 as:

$$\left\lceil \frac{1000}{3} \right\rceil - \left\lceil \frac{100}{3} \right\rceil = 334 - 34 = 300 \qquad (2)$$

Using the same strategy we can determine that $D_4 = 225$, $D_7 = 128$, $D_{12} = 75$, $D_{21} = 43$, $D_{28} = 32$, and $D_{84} = 10$. So in total we have:

$$D_{3 \vee 4 \vee 7} = 300 + 225 + 128 - (75 + 43 + 32) + 10 = 512$$

So there are 512 integers greater than 100 but less than 1000 divisible by 3, 4, or 7.

Grading: 2 marks correct, or even close. If floors are used instead of ceilings still give full marks, assuming the use the correct approach. Be generous is giving part marks as this question is rather tricky.

11. (**2 marks**) How many non-negative integers greater than 313 but less than 2029 are divisible by 13.

Solution:
There are $\lceil \frac{2029}{13} \rceil - \lceil \frac{313}{13} \rceil = 132$ such integers.

Grading: 2 marks for correct answer, part marks if you are off by one.

12. (**10 marks**) Assume we have an alphabet with just the characters $\{\alpha, \epsilon, \gamma, \tau, \omega, \beta, \sigma, \pi\}$. Answer the following questions about this alphabet.

 (a) Allowing repitition, how many words of length 7 start with $\alpha\beta$ or end with σ?
 (b) Allowing repitition how many words can we form of length 4 that contain the sequence $\pi\pi$?
 (c) How many words of length 6 can be formed where each character is unique?
 (d) How many permutations of the letters contain the string $\beta\pi$?
 (e) How many permutations contain the strings $\alpha\beta\gamma$ and $\epsilon\tau\pi$?

Solution:

 (a) First we count how many words start with $\alpha\beta$. In this case the first two characters are fixed, and since repitition is allowed we can choose from among any of 8 characters for the remaining 5 characters so we have $1 \cdot 8 \cdot 8 \cdot 8 \cdot 8 \cdot 8 = 8^5$ words.

 Next we count how many words end with σ. Using the same logic we deteremine there are 8^6 such words.

 Since some words start with $\alpha\beta$ and end with σ we need to ensure these are not excluded. So we apply the inclusion-exclusion principle and calculate the number of such words, which is 8^4 (three of the characters are fixed, four are free). So in total we have: $8^6 + 8^5 - 8^4 = 262144 + 32768 - 4096 = 290816$ such words.

(b) The sequence $\pi\pi$ must appear in one of three possible positions $\pi\pi$–, -$\pi\pi$-, –$\pi\pi$ (the - can be any letter from our alphabet, including an π). Each pattern includes the $\pi\pi$ plus two free characters that can be any one of the eight letters, so there are $1 \cdot 8 \cdot 8 = 64$ words for each pattern. Since there are three patterns we get $64 + 64 + 64 = 192$ total words. However, some of the words are duplicates. There are 8 words with $\pi\pi\pi$- (intersection of $\pi\pi$– and -$\pi\pi$-), and 8 words with -$\pi\pi\pi$ (intersection of -$\pi\pi$- and –$\pi\pi$), and 1 word with $\pi\pi\pi\pi$ (at intersection of all patterns). This is an inclusion/exculsion problem, so we remove $8 + 8 + 1$ for the intersecting area, and add $\pi\pi\pi\pi$ back (in the intersection of all three) for a total of $192 - 17 + 1 = 176$ words.

(c) There are 8 ways to choose the first character, 7 ways to choose the second, and so forth, so there are $8x7x6x5x4x3 = 20160$ ways. Another way to think of this is as a 6-permutation of a set with 8 elements, so we have $P(8, 6) = 20160$.

(d) $P(7, 7)$ since we effectively now have an alphabet of 7 characters where $\beta\pi$ is one of the characters.

(e) $P(4, 4)$ by the same logic as the previous question.

Grading: 2 marks for correct answer. Part marks for partial answers.

13. (**6 marks**) Say we have a group of 10 dogs, 12 cats, and 6 chickens:

(a) How many different groups of 7 animals can we make.

(b) How many different groups of 5 animals have at least 1 cat.

(c) How many different groups of 5 animals have at least one dog, one cat if we can only pick dogs and cats.

Solution:

(a) $\binom{10+12+6}{7} = \binom{28}{7}$.

(b) $\binom{28}{5} - \binom{28-12}{5}$ Total number of ways to pick 5 animals minus the number of ways to pick 5 non-cats.

(c) There are groups of 5 animals, with at least one dog and one cat. So in total we can 4 possible groupings with (1 dog, 4 cats), (2 dogs, 3 cats), (3 dogs, 2 cats), or (4 dogs, 1 cat). We then use the sum rule to add the sizes of these exclusive groups together as follows:
$\binom{10}{1} \cdot \binom{12}{4} + \binom{10}{2} \cdot \binom{12}{3} + \binom{10}{3} \cdot \binom{12}{2} + \binom{10}{4} \cdot \binom{12}{1}$

Grading: 2 marks correct, part marks where progress was made on right track.

14. (**2 marks**) Let n be an integer where $n > 2$. How many bit strings of length n, *or less*, are there that start with 1 and end with 1.

Solution:

Begin by determining how many bit strings of length n start with an 1 and end with an 1. Consider $n = 1$, in this case the bit string "1" both starts and ends with a 1, so there is one such bit string. For $n = 2$ the bit string "11" starts and ends with a 1. For $n = 3$ the bit strings "1*1" qualify, where "*" may be $\{0, 1\}$, so there are $2^1 = 2$ such bit strings. For $n = 4$ the bit strings "1**1" qualify, so there are $2^2 = 4$ such bit strings, and so forth. In general for $n > 2$ there are 2^{n-2} such bit strings.

So for a given value of n how many bit strings are there in total. Let T be the total number of strings over all possible lengths from 1 to n, we can calculate T as follows:

$$\begin{aligned} T &= 1 + 1 + 2^1 + 2^2 + \ldots + 2^{n-2} \\ &= 1 + 2^0 + 2^1 + 2^2 + \ldots + 2^{n-2} \end{aligned}$$

So what does this add up to? Well, dropping the first 1 term we have the summation $2^0 + 2^1 + 2^2 + \ldots + 2^{n-2}$. This is a *geometric series* of the form $\sum_{j=0}^{m} ar^j$ with $m = n - 2$, $a = 1$, and $r = 2$. By Theorem 1 on page 155 of the text we have:

$$\sum_{j=0}^{m} ar^j = \frac{ar^{m+1} - a}{r - 1}$$

substituting our values for m, a and r we get:

$$\begin{aligned}\sum_{j=0}^{n-2}(1)2^j &= \frac{(1)2^{n-1}-1}{2-1} \\ &= 2^{n-1}-1\end{aligned}$$

Now recall that we are not counting bit strings of length 1 so we add 1 we have:

$$T = 1 + 2^{n-1} - 1 = 2^{n-1} \tag{3}$$

Grading:

1 Mark for the number of bit strings of lenght n correct. 1 Mark for figuring out the sum of all such bit strings. Give full marks if they are off by 1 in the summation.