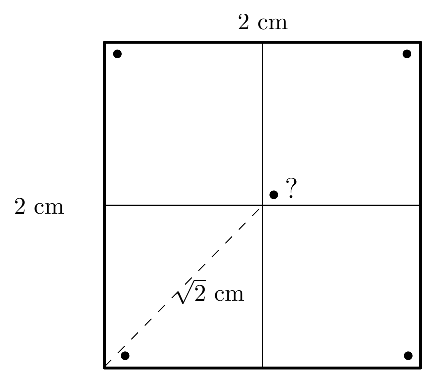
COMP 1805 Discrete Structures
Assignment 4

Due Thursday, June 14th, 2012 at break (before 7:30pm)

Write down your name and student number on **every** page. The questions should be answered in order and your assignment sheets must be stapled, otherwise the assignment will not be marked. Total marks are 30.

1. (**2 marks**) Let S be a 2 cm by 2 cm square. Prove that it is impossible to place five points inside S such that all points are more than $\sqrt{2}$ cm apart (Hint: use the pigeonhole principle).

   Solution:
   A 2 x 2 square can be subdivided into 4 small 1 x 1 cm squares. By the Pythagorean theorem the diagonal of each of these squares has a length of $\sqrt{2}$. Thus the maximum distance between any two points inside (or on the boundary of) one of the small squares is at most $\sqrt{2}$. By the pigeonhole principle since there are four boxes, and five points, some small square must contain two points, and these points are less than or equal to $\sqrt{2}$ cm apart (see Fig **??**).

   Grading:

2. (**2 marks**) How many solutions are there to the inequality

$$2x_1 + 2x_2 + 2x_3 < 32$$

   where x_1, x_2, x_3 are non-negative integers?

   Solution:
   We divide both sides of the equation by two and then introduce a 4th variable x_4 and consider the equation $x_1 + x_2 + x_3 + x_4 = 15$. We can look at x_1, x_2, x_3 and x_4 as being containers for which we have 15 marbles to distribute among them. We must have three dividers, one divider between each container. This means we must find how many distinct ways we can arrange the marbles and dividers. We therefore have $n = 4$ and $r = 15$ so the solution is $C(4 + 15 - 1, 15) = \binom{18}{15}$ or $\binom{18}{3}$.

   Grading:
   2 marks for correct solution and answer. Part marks if 4th variable is introduced but the correct answer is missed.

3. (**2 marks**) There is a video on Youtube of the muppets singing the song MAHNAMAHNA (the spaces have been dropped). How many different unique strings can be formed by rearranging the letters in the title of this song?

Solution:
There are 14 letters in MAHNAMAHNA so there are $10!$ permutations of the letters, but there are a number of duplicates including 4 A's, and 2 of each of { H,M,N }, so the total number of unique strings is $\frac{10!}{4!2!2!2!} = 18900$.

Grading:

4. (**2 marks**) Assume a person is standing at point (3,5) on a discrete grid (all grid coordinates are integers) and assume that they can take steps in either the positive x or y directions (i.e. if they are at point (4,7) they can go to (5,7) or (4,8)). Using this rule, how many different routes are there from (3,5) to position (9,10). Note that two routes are considered different is they differ in any step.

Solution:
To simplify things we will transform the problem from going from (3,5) to (9,10) to that of going from (0,0) to (6,5), since it will involve the same number of steps. Then to go from (0,0) to (6,5) you need to take 6 horizontal, and 5 vertical steps. There are 11 steps in total and 5 of these must be vertical, and the remaining 6 horizontal. Therefore the are $\binom{11}{5}$ or alternately $\binom{11}{6}$ ways to go from (3,5) to (9,10).

Grading:
2 marks for correct solution and answer. 1 mark if you have the idea of dividing path into horizontal, vertical steps. 1/2 if you figure out the transformation.

5. (**6 marks**) Consider a relation R defined on the integers. Determine for the following if the relations are reflexive, symmetric, anti-symmetric, transitive.

   (a) $R = \{(a, b)|a = \frac{b}{2}\}$
   (b) $R = \{(a, b)|ab \geq 0\}$
   (c) $R = \{(a, b)|a > b^2\}$

Solution:

(a)
- Reflexive: The relation is not reflexive since $(1, 1) \notin R$ because $1 \neq 1/2$.
- Symmetric: It is not symmetric since $(1, 2) \in R$ but $(2, 1) \notin R$ since $2 \neq 1/2$.
- Anti-Symmetric: It is anti-symmetric. To show this, we need to prove that $\forall\, x, y \in \mathbb{Z}$, if $(x, y) \in R$ and $(y, x) \in R$, then $x = y$. We will prove this by contradiction. Suppose that $x \neq y$ and $(x, y) \in R$ and $(y, x) \in R$. Since $(x, y) \in R$, this means that $x = y/2$ which means that $y = 2x$. However, $(y, x) \in R$ means that $y = x/2$. But this is a contradiction since we cannot have $y = 2x$ and $y = x/2$ at the same time unless both x and y are zero. But if they're both zero, then $x = y$.
- Transitive: The relation is not transitive since $(2, 4) \in R$ and $(4, 8) \in R$ but $(2, 8) \notin R$.

(b)
- Reflexive: The relation is reflexive. We need to show that $\forall x \in \mathbb{Z},\ (x, x) \in R$. We have $(x, x) \in R$, then $x^2 \geq 0$. But this is true for all integers x.
- Symmetric: It is symmetric. To show this, we need to show that if $(x, y) \in R$ then $(y, x) \in R$. We'll prove this directly. If $(x, y) \in R$, this means that $xy \geq 0$. But this means that $yx \geq 0$ which means that (y, x) is also in R as required.
- Anti-symmetric It is not anti-symmetric since $(1, 2) \in R$ and $(2, 1) \in R$ and $2 \neq 1$.
- Transitive: The relation is not transitive. $(-2, 0) \in R$ since $-2 * 0 \geq 0$ and $(0, 2) \in R$ since $0 * 2 \geq 0$, however $(-2, 2) \notin R$ since $-2 * 2 < 0$.

(c)
- Reflexive: The relation is not reflexive since $(1, 1) \notin R$ because $1 \not> 1^2$.
- Symmetric: It is not symmetric since $(2, 1) \in R$ but $(1, 2) \notin R$ since $1 \not> 2^2$.
- Anti-Symmetric: It is anti-symmetric. To show this, we need to prove that for integers x and y, if $(x, y) \in R$ and $(y, x) \in R$, then $x = y$. We will prove this by contradiction. Suppose that $x \neq y$ and $(x, y) \in R$ and $(y, x) \in R$. Since $(x, y) \in R$, this means that $x > y^2$. However, $(y, x) \in R$ which means that $y > x^2$. This implies that $y^2 > x^4$. But this is a contradiction since we cannot have $x > y^2$ and $y^2 > x^4$ at the same time as this would imply that $x > x^4$ which is not possible for any integer x.

- Transitive: The relation is transitive since if $(x, y) \in R$ and $(y, z) \in R$ we have $x > y^2$ and $y > z^2$. This means that $x > z^4$, which implies that $x > z^2$ since $z^4 > z^2$. Therefore $(x, z) \in R$.

Grading:

6. (**2 marks**) Let A be a set of size 4. How many reflexive relations on A are there?

   Solution:
   There are 16 possible elements ($4x4$) but the 4 diagonals are always part of the any reflexive relation, so we have 12 possibiliites = $2^1 2$.

   Grading:

7. (**4 marks**) Use Warshall's algorithm to find the transitive closures of the following relations on $\{a, b, c, d, e\}$. By use Warshall's algorithm it is meant that you should draw the state of the matrices W_0, W_1, etc.

   (a) $\{(a, d), (b, c), (c, b), (c, d), (d, c), (e, a), (e, e)\}$
   (b) $\{(a, a), (a, c), (b, e), (c, b), (d, e), (e, c)\}$

   Solution:

(a)

$$W_0 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \quad W_1 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix} \quad W_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix} \quad W_3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$W_4 = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad W_5 = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

(b)

$$W_0 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad W_1 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad W_2 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad W_3 = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} \quad W_5 = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

   Grading:
   2 marks for each run of the algorithm worked out properly. If some minor mistake was made that leads to error deduct 1/2 mark, more marks for more major errors (ie. don't get W_0 correct).

8. (**2 marks**) Let R be a reflexive relation defined on the set D. Prove or disprove that R^n is a reflexive relation for all $n \geq 1$.

   Solution:
   We show that R^n is a reflexive relation for all $n \geq 1$ by induction. The base case is for $n = 1$. Since the relation $R = R^1$ is reflexive by definition, the base case holds. The inductive hypothesis states that R^i is reflexive for $1 \leq i \leq k,\ k \geq 1$. We need to prove that R^{k+1} is a reflexive relation. We need to show that $\forall x \in D,\ (x, x) \in R^{k+1}$. Note that $R^{k+1} = R^k \circ R$, Since $(x, x) \in R^k$ and $(x, x) \in R$, we conclude that $(x, x) \in R^{k+1}$ by composition.

   Grading:

9. (**2 marks**) Show that the relation consisting of all (x, y) such that x and y are bit strings of length 5 or more that agree except perhaps in their first five bits is an equivalence relation on the set of all bit strings of length five or more.

Solution:
To prove that this is an equivalence relation it must be *reflexive*, *transitive*, and *symmetric*. Which we do as follows:

- Let x be a bit string of length $n \geq 5$. Any bitstring agrees in all its bits, so x agrees in all bits (if any) > 5 so $(x, x) \in R$, and R is *reflexive*.
- Assume for simplicity that x and y are bitstrings with length $n \geq 6$. Let $b_0b_1b_2b_3b_4b_5b_6 \ldots b_n$ be the bits of a bitstring. If $(x, y) \in R$ then x and y agree in bits $b_6 \ldots b_n$, hence by the definition of the relation $(y, x) \in R$. Thus R is *symmetric*.
- Let x,y,z be bitstrings of length $n \geq 6$ and assume $(x, y) \in R$ and $(y, z) \in R$. Note that x and y have the same length, otherwise they couldn't agree in all bits $b_6 \ldots b_n$. Likewise y and z have the same length. Furthermore x and y agree in bits $b_6 \ldots b_n$, as do y and z, so x and z must agree in bits $b_6 \ldots b_n$. So $(x, z) \in R$ and the relation is *transitive*.

One detail that has been omitted is bit strings of length exactly five. Such bit strings may differ in their bits, but the relation states that any such bit string is equivalent. So all bit strings of length exactly five form a single equivalanence class since they *agree* in all bits $b_6 \ldots b_n$ (*eg.* they agree in the sense that they don't include these bits).

Grading:

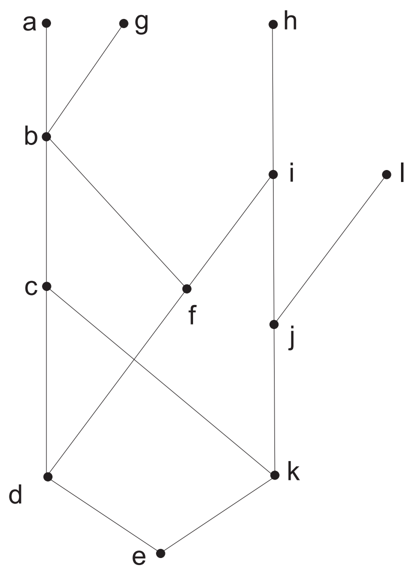
10. (**6 marks**) Consider the Hasse diagram shown below on the set $\{a, b, c, d, e, f, g, h, i, j, k, l\}$. Answer the following questions (when listing elements of a solution set please list them in alphabetical order if you can do so).

(a) What are the maximal elements.
(b) What are the minimal elements.
(c) Is there a greatest element.
(d) Is there a least element.
(e) Find all upper bounds for $\{d, e, k\}$.
(f) Find all lower bounds for $\{a, g, h\}$.
(g) Find the greatest lower bound for $\{a, g, h\}$ if it exists.
(h) Use topological sort (showing your steps as in class and in the textbook in Figures 10 and 11 of Chapter 8) to generate a compatible total order on the set.

Solution:

(a) Maximal elements: $\{a, g, h, l\}$.
(b) Minimal elements: $\{e\}$.
(c) Greatest element: none.
(d) Least element: $\{e\}$.
(e) Upper bounds for $\{d, e, k\}$: $\{a, b, c, g, h, i\}$.
(f) Lower bounds for $\{a, g, h\}$: $\{d, e, f, k\}$.
(g) Greatest lower bound $\{a, g, h\}$: none. Note that students often think that f should be a greatest lower bound in this case. However k is a lower bound which is not comparable to f and as such f cannot be the greatest lower bound.
(h) Topological sort: There are numerous possible valid solutions, see figure below for one which produces the total order $e \prec k \prec d \prec c \prec j \prec l \prec f \prec b \prec a \prec i \prec g \prec h$.

Grading:
Parts a through g are worth 1/2 mark each. Part h (topological sort) is worth 2.5 marks total. 1 mark for a valid total order and 1.5 marks for showing steps correctly


a
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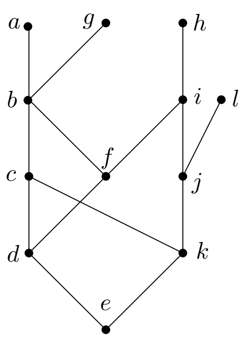
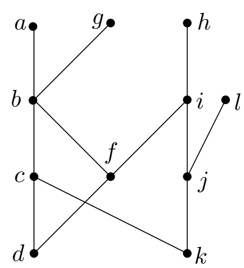
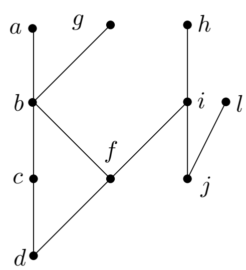


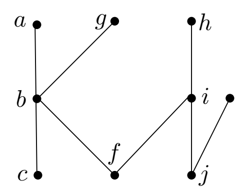
1. Remove e



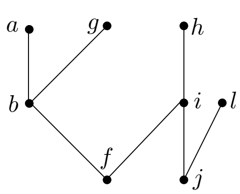
2. Remove k



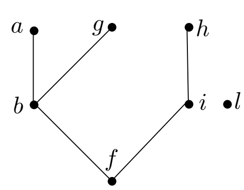
3. Remove d



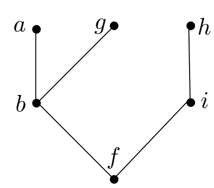
4. Remove c



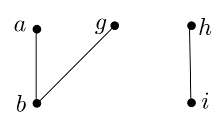
5. Remove j



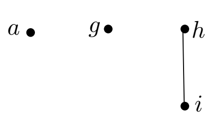
6. Remove l



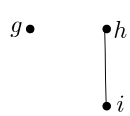
7. Remove f



8. Remove b



9. Remove a



10. Remove i

11. Remove g

12. Remove h