LECTURE 10

Relations (Properties Continued).

Last class we saw relations were formed by a subset of the Cartesian product $A \times B$ of two sets (or by a set onto itself) in which case we call the relation a relation on A.

For relations on a set we defined the following properties

1. reflexive : if $(a,a) \in R$ for every $a \in A$.
2. symmetric : if $(b,a) \in R$ whenever $(a,b) \in R$ for all $a,b \in A$.
3. antisymmetric : if $(a,b) \in R$ and $(b,a) \in R$ then $a = b$. A.

We now consider one more property of a relation on a set.

Transitivity

A relation R on a set A is transitive if $((a,b) \in R$ and $(b,c) \in R) \rightarrow (a,c) \in R$. for all $a,b,c \in A$.

Ex: On $\mathbb{Z}$

$R_1 = \{ (a,b) \mid a \leq b \}$
$R_2 = \{ (a,b) \mid a > b \}$
$R_3 = \{ (a,b) \mid a = b \}$
$R_4 = \{ (a,b) \mid a = b \text{ or } a = -b \}$

are all transitive.

For example R_1 is transitive since if $(a,b) \in R_1$ then $a \leq b$. If $(b,c) \in R_1$ then $b \leq c$, thus $a \leq c$ so $(a,c) \in R_1$.

The Relations

$R_5 = \{(a,b) \mid a = b+1\}$
$R_6 = \{(a,b) \mid a+b \leq 3\}$
are not transitive.

For example $(2,1) \in R_5$ and $(1,0) \in R_5$, but $(2,0) \notin R_5$.
$(2,1) \in R_6$ and $(1,2) \in R_6$, but $(2,2) \notin R_6$.

Is the "divides" relation transitive?

Suppose a divides b and b divides c, then

$b = a \cdot k$, and $c = b \cdot l$ for integers k, l.

$b = a \cdot k$
$c = b \cdot l$ $\therefore$ $c = a \cdot \underbrace{k \cdot l}_{\text{an integer}}$ so a divides c.

Thus the divides relation is transitive.

Combining Relations

Relations are sets, so they can be combined just like sets can be combined.

Ex: Let $A = \{1,2,3\}$ and $B = \{1,2,3,4\}$

The relations $R_1 = \{(1,1), (2,2), (3,3)\}$ and
$R_2 = \{(1,1), (1,2)\ (1,3), (1,4)\}$ from A to B can be combined as follows:

$R_1 \cup R_2 = \{(1,1), (1,2)\ (1,3), (1,4)\ (2,2)\ (3,3)\}$
$R_1 \cap R_2 = \{(1,1)\}$
$R_1 \setminus R_2\ (R-R_2) = \{(2,2)\ (3,3)\}$
$R_2 - R_1\ (R_2 \setminus R_1) = \{(1,2)\ (1,3), (1,4)\}$

Ex: Let $R_1 = \{(a,b) \mid a<b\}$ and $R_2 = \{(a,b) \mid a>b\}$ be relations defined on the set of reals ($\mathbb{R}$).

$$R_1 \cup R_2 = \{(a,b) \mid a<b \text{ or } a>b\}$$
$$\{(a,b) \mid a \neq b\}$$

$$R_1 \cap R_2 \quad \{(a,b) \mid a<b \text{ and } a>b\} = \emptyset$$

$$\left.\begin{array}{l} R_1 - R_2 = R_1 \\ R_2 - R_1 = R_2 \end{array}\right\} \text{ since the relations include no common elements.}$$

Remember that relations are much like functions, so it makes sense to talk about thier composition too:

Let R be a relation from set A to set B. Let S be a relation from set B to set C. The <u>composite</u> of R and S is the relation consisting or ordered pairs (a,c) where $a \in A$, $c \in C$ and there exists an element $b \in B$ such that $(a,b) \in R$ and $(b,c) \in S$. Write it $S \circ R$.

Ex: $R = \{(1,1), (1,4), (2,3), (3,1), (3,4)\}$ - from $\{1,2,3\}$ to $\{1,2,3,4\}$.
$S = \{(1,0), (2,0), (3,1), (3,2), (4,1)\}$ - from $\{1,2,3,4\}$ to $\{0,1,2\}$

$$S \circ R = \{(1,0), (1,1), (2,1), (2,2), (3,0), (3,1)\}$$

One special case of composition occurs when you compose a relation with itself:

Ex: Let $R = \{(a,b) \mid a \text{ is parent of } b\}$ be defined on the set of all people.

Then $R \circ R = \left\{(a,c) \,\middle|\, \begin{array}{l} \exists \text{ person } b \text{ such that } a \text{ is parent} \\ \text{of } b \text{, and } b \text{ is the parent of } c \end{array}\right\}$

So $R \circ R = \{(a,c) \mid a \text{ is the grandparent of } c\}$

This produces another relation, which you can the compose again:

$R \circ R \circ R = \left\{(a,d) \,\middle|\, \begin{array}{l} a \text{ is a great-grandparent} \\ \text{of } d \end{array}\right\}$

We write $R \circ R = R^2$, $R \circ (R \circ R) = R^3$, etc.

In general we have the recursive definition

$$R^1 = R \qquad R^{n+1} = R^n \circ R$$

Ex: $R = \{(1,1), (2,1), (3,2), (4,3)\}$

$R^1 = R$

$R^2 = \{(1,1), (2,1), (3,1) (4,2)\}$

$R^3 = \{(1,1), (2,1), (3,1), (4,1)\}$

$R^4 = R^3$

$R^n = R^3$

A relation on a set A is transitive iff $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$ (Thm 1 p. 527).

<u>Prove</u> (If $R^n \subseteq R$ ~~then~~ $\rightarrow$ relation R on A is transitive).

Assume $R^n \subseteq R$ for $n = 1, 2, 3$.

Eg. $R^2 \subseteq R$.

1. If $(a,b) \in R$ and $(b,c) \in R$, then by definition of a composition of relations $(a,c) \in R^2$.

2. Because $R^2 \subseteq R$, these means $(a,c) \in R$ so R is transitive.

Prove (Relation R on A is transitive $\longrightarrow$ $R^n \subseteq R$)

Prove by induction.

<u>Basis Step</u>: $n = 1$. If R is transitive, then $R^1 = R$ is transitive.

<u>Inductive Hypothesis</u>: $(R^k \subseteq R) \rightarrow (R^{k+1} \subseteq R)$

Assume $(a,b) \in R^{k+1}$

Since $R^{k+1} = R^k \circ R$, there is an element $x \in A$ such that $(a,x) \in R$ and $(x,b) \in R^k$.

[This is due to the definition of composition of relations]

By the inductive hypothesis $(x,b) \in R$

Since R is transitive and $(a,x), (x,b) \in R$, $(a,b) \in R$. Thus $R^{k+1} \subseteq R$.

How do we Represent Relations (Assume Binary Relations).

- List all elements $\{ (a,b) (b,c) (c,d) \}$.
- Via a formula $\{ (a,b) \mid a > b \}$

Matrices & Directed graphs can also be used.

Matrix Representation (Zero-one matrix).

Set $A = \{ a_1, a_2, \ldots a_m \}$

$B = \{ b_1, b_2, \ldots b_n \}$

Matrix $M_R = [m_{ij}]$ where

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

Ex:

$A = \{1,2,3\}$ $B = \{1,2\}$

$R = \{ (a,b) \mid a \in A, b \in B \text{ and } a > b \}$

$\{ (2,1), (3,1), (3,2) \}$

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \quad A$$

B

Matrices can show tell us if a relation is reflexive, symmetric, antisymmetric.

For relations on a set.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Reflexive. Symmetric Antisymmetric

Representing Relations Using Diagraphs

A directed graph, or digraph, consists of a set V of vertices together with a set E of ordered pairs of elements of V called edges.

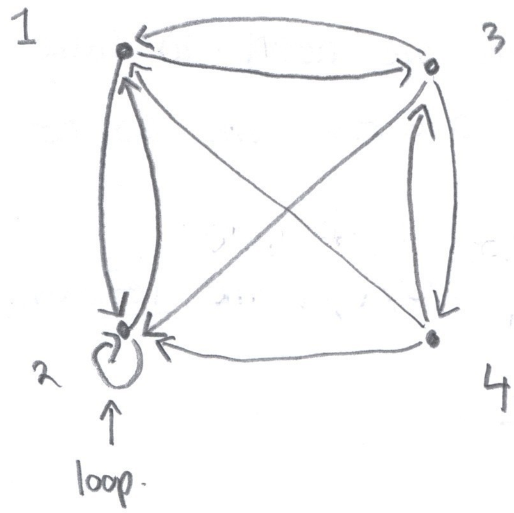
For an edge (a,b). a is the initial vertex of the edge and b is the terminal vertex of the edge.

A relation of set A has

- set A as vertices
- ordered pairs $(a,b) \in R$ as its edges

Ex: $A = \{1,2,3,4\}$

$R = \{(1,2),(1,3),(2,1),(2,2),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)\}$



Relation Properties in Diagraphs

- every vertex has a loop – reflexive.
- every edge between distinct vertices has an corresponding edge in opposite direction – symmetric
- no pair of vertices are joined by edges with opposite directions – antisymmetric.

Closures of Relations (Section 8.4)

Sometimes a relation does not have some property that we would like: eg. reflexivity, symmetry, transitivity.

How do we add elements to our relation to guarantee the desired property – ideally adding as few new elements to R (the relation) as possible.

Reflexive Closure

Suppose we had a relation R on set A and want to make it reflexive. Then we need to make sure (a,a) is in the relation for all $a \in A$. We do not want to add anything extra.

Define $\Delta = \{(a,a) \mid a \in A\}$. The <u>reflexive closure</u> of R is $R \cup \Delta$.

Ex: What is the reflexive closure of $R = \{(a,b) \mid a<b\}$ on the set of integers.

$$R \cup \Delta = \{(a,b) \mid a<b\} \cup \{(a,a) \mid a \in \mathbb{Z}\}$$
$$= \{(a,b) \mid a \leq b\}$$

Symmetric Closure

Want to ensure $(b,a) \in R$ whenever $(a,b) \in R$. Same idea as before. Define $R^{-1} = \{(b,a) \mid (a,b) \in R\}$, then the symmetric closure of R is $R \cup R^{-1}$.

Ex: Symmetric closure of $R = \{(a,b) \mid a>b\}$ on positive integers is $R \cup R^{-1} = \{(a,b) \mid a>b\} \cup \{(b,a) \mid b>a\}$

$$= \{(a,b) \mid a \neq b\}$$

Transitive Closure

Consider the relation $R = \{(1,3), (1,4), (2,1), (3,2)\}$ on $\{1,2,3,4\}$

It is not transitive since it is missing $(1,2)$, $(2,3)$ $(2,4)$ and $(3,1)$.

Lets add them:

$R = \{(1,2)\ (1,3)\ (1,4), (2,1)\ (2,3)\ (2,4), (3,1)\ (3,2)\}$

Still not transitive $(3,1)$ $(1,4)$ but not $(3,4)$

It seems we will need a bit more work

Idea 💡: If we compose R with itself. Then we get the elements (a,c) where $(a,b) \in R$ and $(b,c) \in R$ for some b. We need these for transitivity, so add them to R. As we saw in our previous attempt, this might be enough, so we <u>repeat</u>.

How many times do we repeat? This will depend of course on how many intermediate elements we might find.

$$(a_0, a_1), \quad (a_2, a_3), \quad (a_3, 4) \quad \ldots \quad (a_{n-1}, a_n) \qquad n = |A|$$

We must continue until we can guarantee $(a_0, a_n) \in R'$. This may take 'n' steps, since at each composition we we may only add one new element (guaranteed).

Thus, the transitive closure of R is:

$$R \cup R^2 \cup R^3 \cup \ldots \cup R^n$$

Ex: $R = \{(1,1), (1,3), (2,2), (3,1), (3,2)\}$ on $\{1,2,3\}$

$R^1 = R$

$R^2 = \{(1,1)\ (1,2)\ (1,3), (2,2)\ (3,1)\ (3,2)\ (3,3)\} = R^3$

So the transitive closure is:

$$R \cup R^2 \cup R^3 = R \cup R^2 \quad \text{or}$$

$$\{(1,1)\ (1,2)\ (1,3)\ (2,2)\ (3,1)\ (3,2)\ (3,3)\}$$

There is another way to look at this problem, and that is using the matrix representation. Given matrix representations M_R and M_S for R and S the matrix representation for $S \circ R$ is $M_R \odot M_S$, where $\odot$ denotes the 'join' operation, this is identical to matrix multiplication, except that any non-zero entries are simply written as 1.

Ex: Matrix multiplication if need be:

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad M_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$M_R \odot M_S \quad \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So in our example (for transitive closure).

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \qquad M_{R^2} = M_R \odot M_R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$M_{R^3} = M_{R^2} \odot M_R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

And the transitive closure is the union of these relations, where the union of two 0-1 matrices corresponds to applying the 'V' operation to each pair of entries.

Thus the transitive closure is:

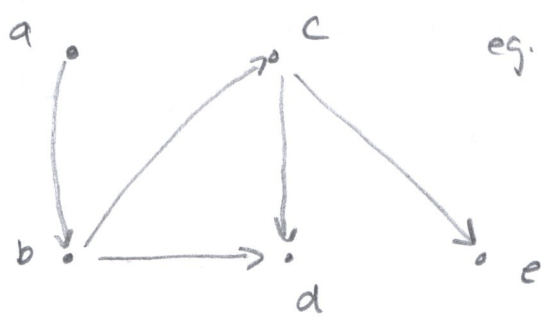
$$\underset{R}{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}} \vee \underset{R^2}{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}} \vee \underset{R^3}{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{R^*}{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}}$$

This resulting matrix (relation) is called the <u>connectivity relation</u> R^*, we can express it as

$$R^* = \bigcup_{n=1}^{\infty} R^n$$

Note we can go to infinity because after R^n where $n = |A|$ we no longer worry about adding new elements. ie. $R^n = R^{n+1} = R^{n+2}$ and so on. THIS algorithm takes $O(n^4)$ time.

If we think of our directed graph example, R^*, contains all pairs such that there is a <u>path</u> of length at least one from a to b in R.



eg. there is path of length 3 from (a,e) in this graph so $(a,e) \in R^*$, but not an element of R

a, <u>b, c</u>, e is the path, b, c are <u>interior</u> vertices on this path.

Warshalls Algorithm

If we look at the relation as a graph, we can build up longer and longer paths between elements. To do this, we allow more and more 'interior vertices' along the path.

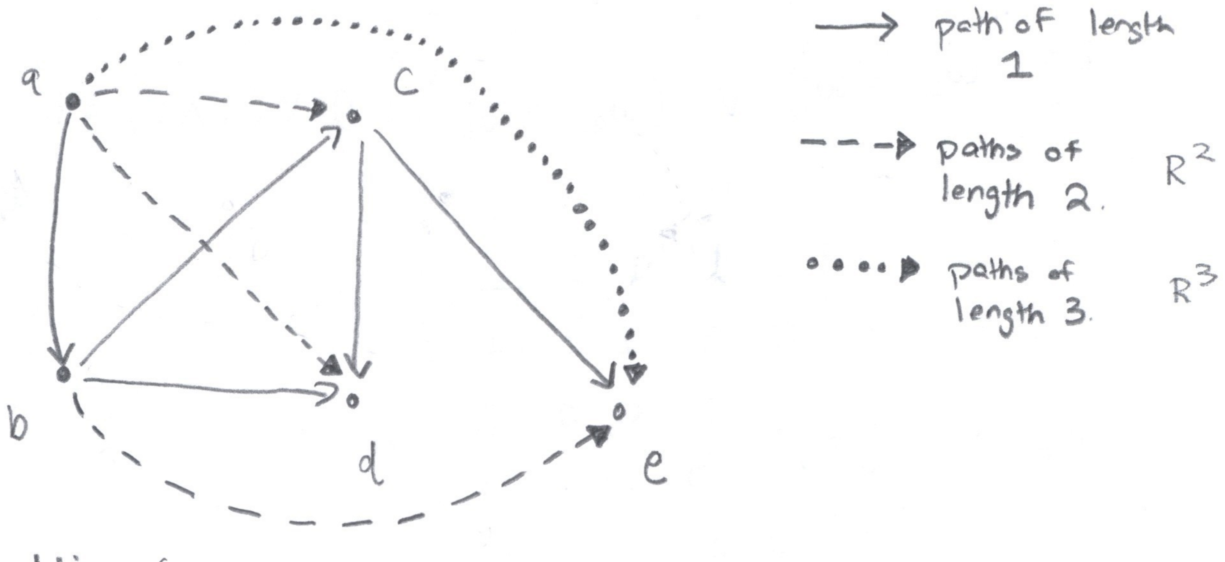
More and more interior vertices correspond to taking more and more compositions.

Idea

- start with the relations initial graph. (vertices connected by paths of length 1)
- add link between vertices that have a path of length 2
- " " " " " " " " " 3
- " " " " " " " " " 4

and so on.

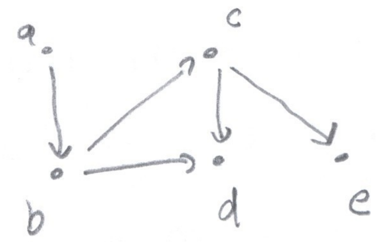
Ex



Key: Note that adding (a,e) is done by adding (c,e) to (a,c).

Warshalls algorithm operates on matrices. Let $W_0 = R$,

$$W_0 = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

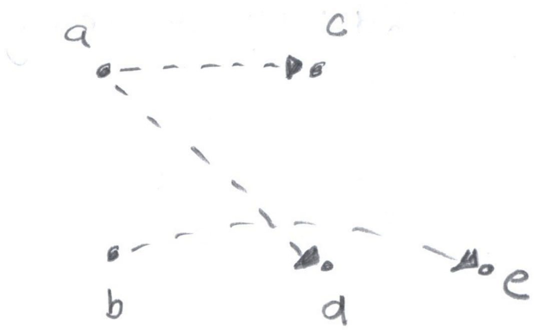


We then construct a sequence of matrices, using the following rule for updating:

$$w_{ij}^{[k]} = w_{ij}^{[k-1]} \vee \left(w_{ik}^{[k-1]} \wedge w_{jk}^{[k-1]} \right)$$

For example.

$$W_1 = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & \underline{1} & \underline{1} & 0 \\ 0 & 0 & 1 & 1 & \underline{1} \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$



Eg: $w_{ac}^{1} = w_{ac}^{0} \vee \left(w_{ab}^{0} \wedge w_{bc}^{0} \right) = 0 \vee (1 \wedge 1) = 1$

$$W_2 = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & \underline{1} \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Eg $w_{ae}^{2} = w_{ae}^{1} \vee \left(w_{ac}^{1} \wedge w_{ce}^{1} \right) = 0 \vee (1 \wedge 1) = 1$

To complete Warshalls algorithm we would go on to compute W_3, W_4 and $W_5 = W_{R^*}$, but in this case we would add no more elements to the relation, so we wont bother listing all of the steps.

The algorithm's pseudo code.

Warshall (M_R : $n \times n$ zero one matrix)

$W \leftarrow M_R$

for $k = 1$ to n

for $i = 1$ to n

for $j = 1$ to n

$w_{ij} = w_{ij} \vee (w_{ik} \wedge w_{kj})$ } n steps } n } n - $O(n^3)$.

return W

The running time is $O(n^3)$ which is a factor n better than the matrix multiplication technique we we shown first.