LECTURE 11

Equivalence Relations

We intuitively know what it means to be 'equivalent' and some relations satisfy this intuition, while others do not.

Consider the "=" symbol. This defines a relation – we normally write $1 = 1$, for example, but we can also define:

$$R = \{ (a,b) \mid (a=b) \} \text{ on } \mathbb{Z}.$$

and write $(1,1) \in R$ or $1R1$.

Why do we say the equals relation expresses equivalence?

- anything is equivalent to itself (reflexive)
- $a = b \rightarrow b = a$ (symmetric)
- $a = b \wedge b = c \rightarrow a = c$ (transitive).

Other examples include logical equivalence, set equivalence and many others

Defn: A relation is an equivalence relation if it is reflexive, symmetric, and transitive.

Ex. Let $R = \{ (a,b) \mid a = b \text{ or } a = -b \}$ on $\mathbb{Z}$.

Is R an equivalence relation?

① Symmetric: Assume $(a,b) \in R$, then $a = b$ or $a = -b$.

If $a = b$, then $b = a$ so $(b,a) \in R$.

If $a = -b$, then $b = -a$ so $(b,a) \in R$

② Reflexive: $a = a$ so $(a,a) \in R$.

③ Transitive: Assume $(a,b) \in R$ and $(b,c) \in R$.

Then $a = b$ or $a = -b$, and $b = c$ or $b = -c$.

If $a = b$ and $b = c$ then $a = c$ so $(a,c) \in R$.

If $a = b$ and $b = -c$ then $a = -c$ so $(a,c) \in R$.

If $a = -b$ and $b = c$ then $-b = -c$ so

$a = -b = -c$ so $(a,c) \in R$

If $a = -b$ and $b = -c$ then $-b = c$ so

$a = -b = c$ so $(a,c) \in R$

Thus R is symmetric, reflexive and transitive and therefore it defines an equivalence relation.

Ex: $R = \{ (a,b) \mid m \text{ divides } a-b \}$ for some positive integer m on $\mathbb{Z}^+$.

This relation is called 'Congreunce modulo m' (we didn't cover this topic, but it is a good example of an equivalence relation).

Recall that 'c divides d $\iff$ d = k·c for some $k \in \mathbb{Z}$.

① Reflexive: m divides a−a, $a-a=0$? $0=0\cdot m$ so yes.

② Symmetric: Assume $(a,b) \in R$, So m divides a−b, thus $a-b = k\cdot m$ for some $k \in \mathbb{Z}$. So is $(b,a) \in R$?
Well, $b-a = (-k)\cdot m$, so m divides b−a.
So $(b,a) \in R$ and R is symmetric.

③ Transitive: Assume $(a,b) \in R$ and $(b,c) \in R$. To prove R is transitive we mush show $(a,c) \in R$. We have $(a-b) = k_1 m$ and $(b-c) = k_2 m$. Since $a-c = (a-b)+(b-c)$ we have $a-c = k_1 m + k_2 m = m(k_1+k_2)$ where k_1+k_2 is an integer so $(a,c) \in R$, and R is transitive.

Ex: R is a relation on the set of strings of English letters such that $(a,b) \in R$ if and only if a and b have the same length. Is it an Equivalence relation?

Reflexive: Yes, any string has the same length as itself.

Symmetric: If $(a,b) \in R$ then $|a| = |b|$, so $|b| = |a|$ and $(b,a) \in R$.

Transitive: If $(a,b) \in R$ and $(b,c) \in R$ then $|a| = |b| = |c|$ So $(a,c) \in R$.

Note, I am using $|a|$ to denote the length of string a.

Ex: Is the divides relation an equivalence relation?

Reflexive: $a \mid a$ so $a = 1 \cdot a$

Transitive: If $a \mid b$ and $b \mid c$ then $b = k_1 \cdot a$ and $c = k_2 b$, so $c = k_2 \cdot (k_1 \cdot a) = k_1 k_2 a$, so $a \mid c$.

Symmetric: $2 \mid 4$ but $4 \nmid 2$, so R is not symmetric.

$\therefore$ the divides relation is not an equivalence relation.

Ex: Is $R = \{(a,b) \mid |a-b| < 1\}$ on $\mathbb{Z}$ an equivalence relation.

Reflexive: $a - a = 0 < 1$

Symmetric: If $(a-b) \in R$ then $|a-b| < 1$ so $|b-a| < 1$ also. So $(b,a) \in R$.

Transitive: No! $x = 2.8$ $y = 1.9$ $z = 1.4$

$|x-y| < 1$ so $(x,y) \in R$

$|y-z| < 1$ so $(y,z) \in R$

$|x-z| = 1.4 > 1$

so $(x,z) \notin R$

NOT transitive, NOT equivalence relation.

Equivalence Classes

Equivalence relations naturally partition the elements of the set thy are defined on into several classes.

Ex: Let $R = \{(a,b) \mid$ a and b received the same grade in COMP 1805$\}$
The domain is the set of students in COMP 1805.

- It should be clear R is an equivalence relation, it is reflexive, symmetric and transitive.
- it also partitions the students into classes:

A+ students
A students
A- students, etc.

Defn: Let R be an equivalence relation on the set A. The set of all elements related to $a \in A$ is called the equivalence class of a, and is denoted $[a]_R$ or $[a]$ when R is clear from the context.

So: $$[a]_R = \{ b \mid (a,b) \in R \}.$$

If $b \in [a]_R$ then b is called a representative of the equivalence class (any member of the equivalence class can be choosen).

Ex: Recall $R = \{(a,b) \mid a = b \text{ or } a = -b\}$ is an equivalence relation.
An integer is equivalent to itself and its negative.

$[a] = \{a, -a\}$

So $[1] = \{1, -1\}$ $\quad [2] = \{2, -2\}$ ... $\quad [0] = \{0\}$

Ex: Let $R = \{(a,b) \mid a \text{ and } b \text{ have same first 3 bits}\}$, where the domain is the set of bitstrings of length 6. This is an equivalence relation, what are the equivalence classes.

$[000000] = \{000000, 000100, 000110, \text{etc} \ldots\}$
$[001000]$
$[010000], [011000], [100000], [101000], [110000], [111000]$

Note, these bit strings are 'representatives' of the 8 equivalence classes for this relation.

Partitions

Recall the grades example, where the equivalence classes where the students who achieved the same grade. Note that this is a partition: two equivalence classes are either equal or disjoint, and the union of all equivalence classes is the original set.

This can work in the other direction too. Given a partition of a set A, we can always construct an equivalence relation R on A with those equivalence classes.

Ex: Partition $A = \{1,2,3,4,5,6\}$ into $A_1 = \{1\}$, $A_2 = \{2,3\}$ and $A_3 = \{4,5,6\}$. Define an equivalence relation on A with equivalence classes A_1, A_2, A_3.

$R_1 = \{(a,b) \mid a,b \in A_1\} = \{(1,1)\}$
$R_2 = \{(a,b) \mid a,b \in A_2\} = \{(2,2)\ (2,3)\ (3,3)\ (3,2)\}$ →
$R_3 = \{(a,b) \mid a,b \in A_3\} = \{(4,4)\ (4,5)\ (4,6)\ (5,4)\ (5,5)\ (5,6)\ (6,4)\ (6,5)\ (6,6)\}$

Then we can define R in terms of R_1 R_2 R_3 as follows:

$$R = R_1 \cup R_2 \cup R_3$$

Partial Orders

A relation R on a set S is a partial ordering or partial order if it is:

- reflexive
- transitive, and
- antisymmetric.

A set S, together with a partial order R is called a partially ordered set or poset and is denoted (S,R). Members of S are called elements of the poset.

Ex: The relation $\geq$ on the set of integers $\mathbb{Z}$ is a partial ordering $(\mathbb{Z}, \geq)$.

Reflexive: $a \geq a$ for all a.

Antisymmetric: If $a \geq b$ and $b \geq a$ then $a = b$.

Transitive: If $a \geq b$ and $b \geq c$ then $a \geq c$.

So $\geq$ is a partial ordering on $\mathbb{Z}$, so $(\mathbb{Z}, \geq)$ is a poset.

What about the relation $>$ on $\mathbb{Z}$?

No. It is not reflexive $a \not> a$.

Ex: We previously saw that the division relation $|$ on $\mathbb{Z}^+$ is reflexive and transitive. Is $(\mathbb{Z}^+, |)$ a poset.

Antisymmetric?

If $a \mid b$ and $b \mid a$ then

$$b = k_1 \cdot a \qquad a = k_2 \cdot b$$

So $b = k_1 \cdot k_2 \cdot b$, so that $k_1 k_2 = 1$. Since $k_1, k_2 \in \mathbb{Z}^+$ $k_1 = k_2 = 1$ so $b = a \cdot k_1 = a$. So "$|$" is antisymmetric and $(\mathbb{Z}^+, |)$ divisibility defines a partial order and this is a poset.

Def$^{\underline{n}}$: The elements a, b of a poset $(S, \underbrace{\preccurlyeq}_{\text{new notation}})$ are <u>comparable</u> if either $a \preccurlyeq b$ or $b \preccurlyeq a$.

When neither is true, they are said to be incomparable.

This notation was developed because we cannot always compare two elements in a poset.

<u>Example</u>: Given the powerset of a set S, $P(S)$. Then $(P(S), \subseteq)$ defines a partial ordering (say $S = \{1,2,3,4\}$).

It is reflexive: $\{1\} \subseteq \{1\}$ $\{1,2\} \subseteq \{1,2\}$ etc.

It is transitive: $\{1\} \subseteq \{1,2\} \subseteq \{1,2,3\}$ so $\{1\} \subseteq \{1,2,3\}$.

It is antisymmetric: If $A_1 \subseteq P(S)$ and $A_2 \subseteq P(S)$ and $A_1 \subseteq A_2$ then if $A_2 \subseteq A_1$ we must have $A_1 = A_2$.

However, there are some sets in the powerset that are not comparable under the $\subseteq$ relation.

eg. $\{1,3\} \not\subseteq \{2,4\}$ $\quad \{2,4\} \not\subseteq \{1,3\}$.

So if $a = \{1,3\}$ and $b = \{2,4\}$ then neither $a \preccurlyeq b$ or $b \preccurlyeq a$. So a and b are uncomparable.

Ex: In the "$|$" relation, $5 \nmid 7$ and $7 \nmid 5$, so these elements are not comparable. However, $2 \mid 4$ so 2 and 4 are comparable.

Ex: In the $\leq$ relation, we always have $a \leq b$ or $b \leq a$ for every pair of elements (a,b), so every pair of elements is comparable.

This 'uncomparability' of some pairs of elements is why we use the term 'partial' orderings.

Sometimes all elements are comparable (such as for $\leq$), for these cases we have the following definition:

Defn: If $(S, \preccurlyeq)$ is a poset and every two elements of S are comparable, S is called a <u>totally ordered</u> (or <u>linearly ordered</u>) set) and $\preccurlyeq$ is called a total or linear order. A totally ordered set is also called a <u>chain</u>.

Ex: $(\mathbb{Z}, \leq)$ is a chain or total order. It is a poset and all elements are comparable.

Ex: $(\mathbb{Z}^+, |)$ is a poset, as we saw before, but not a chain, as some elements are not comparable.

Note: In the text don't worry about Def$\underline{n}$ 4 and Theorem 1 (p. 568) since we didn't cover well ordered sets.

Hasse Diagrams

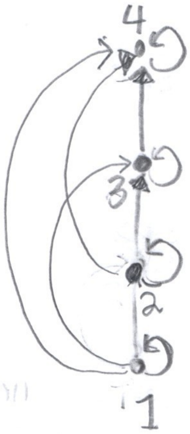
Recall the directed graph (digraph) representation we previously employed to represent a partial order. In representing a poset in this manner we end up drawing alot of lines that we don't need, since we know a poset is:

> reflexive: we can omit the self loops

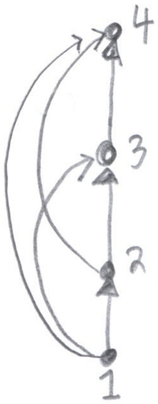
> transitive: we can omit (a,c) if we have (a,b) and (b,c).

So what is the simplest diagram we can use to completely represent a poset.

Example: Consider the directed graph for $(\{1,2,3,4\}, \leq)$, which is a poset.

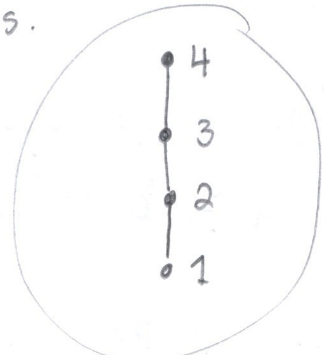


Since we know a poset has self-loops we can remove them →



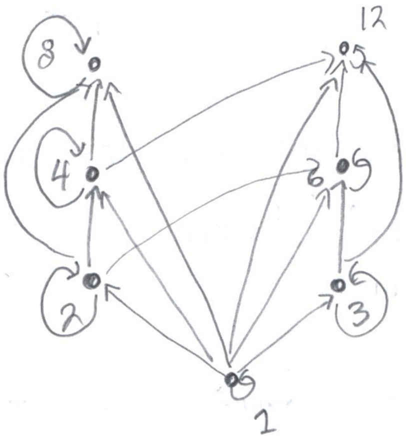
Since it is transitive remove edges that can → be represented by longer paths

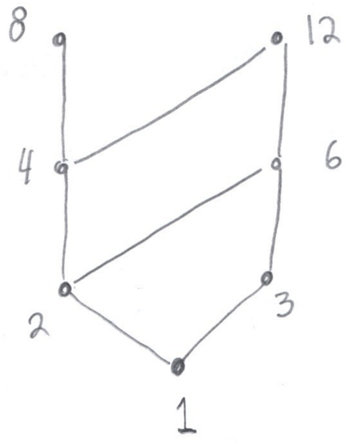
Finally, arrange edges so that the initial vertex is below the terminal vertex and remove the arrows.



— Hasse diagram for $(\{1,2,3,4\}, \leq)$

Ex: Draw the Hasse diagram for $(\{1,2,3,4,6,8,12\}, \mid)$





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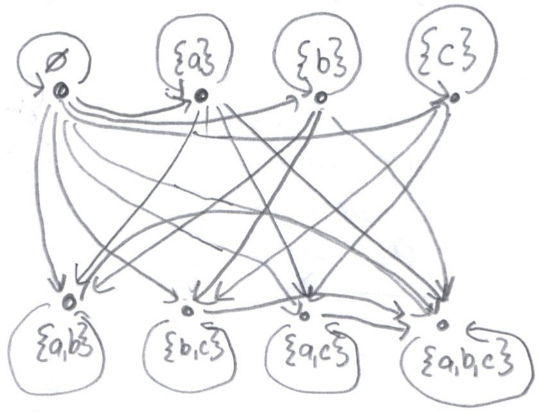
Ex: Hasse diagram for $(P(\{a,b,c\}), \subseteq)$



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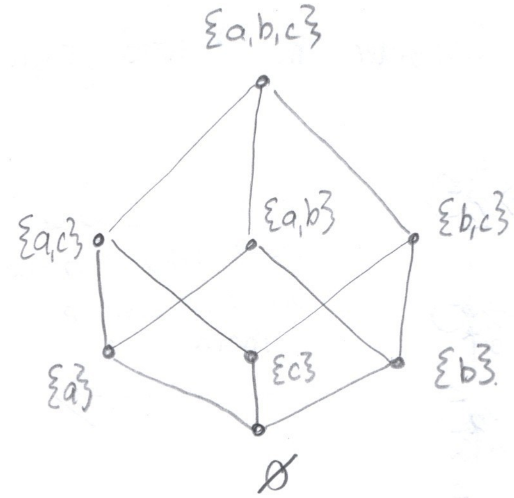

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Another benefit of Hasse diagrams is that it is easier to pick out certain special elements.

An element is <u>maximal</u> in the poset $(S, \preccurlyeq)$ if there is no $b \in S$ with $a \prec b$ (ie $a \preccurlyeq b$, but $a \neq b$)

An element is <u>minimal</u> in the poset $(S, \preccurlyeq)$ if there is no $b \in S$ with $b \prec a$.

In the Hasse diagram the maximal elements). are at the top and the minimum elements are at the bottom

Ex: In $(\{1,2,3,4,6,8,12\}, \mid)$ 8, 12 are maximal, 1 is minimal.

Ex: In $(P(\{a,b,c\}, \subseteq)$ $\{a,b,c\}$ is maximal, $\varnothing$ is minimal.

Sometimes there is an element greater than every other element.
If $b \preceq a$ for all $b \in S$, then a is the greatest element of $(S, \preceq)$.

If $a \preceq b$ for all $b \in S$, then a is the least element of $(S, \preceq)$.

If they exist, both the greatest and least elements are unique.

Ex: In $(\{1,2,3,4,6,8,12\}, |)$ 1 is the least element and there is no greatest element. Since $8 \nmid 12$ and $12 \nmid 8$.

Ex: In $(P(\{a,b,c\}), \subseteq)$, $\emptyset$ is the least element and $\{a,b,c\}$ is the greatest.

Sometimes we want to bound a subset of the poset.

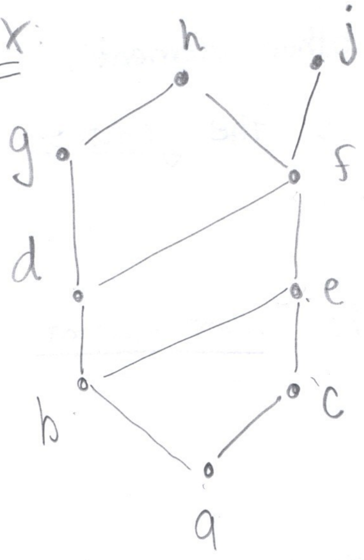
For example, given a poset $(S, \preceq)$ and a subset $A \subseteq S$:

> if $u \in S$ such that $a \preceq u$ for all $a \in A$, then u is called an upper bound of A.

> if $l \in S$ such that $l \preceq a$ for all $a \in A$, then l is called an lower bound of A.

Ex:

Ex:



Given the poset represented by the Hasse diagram shown.

① Upper bounds of $A = \{a, b, c\}$ are
e, f, h, j
But not g since $c \not\leq g$.

② Lower bounds of $\{a, b, c\}$ is a (since $a \leq a$).

③ Upper bounds of $\{j, h\}$ is: NONE $\emptyset$

④ Lower bounds of $\{j, h\}$ is a, b, c, d, e, f (again g is not included since $g \not\leq j$

⑤ Upper bounds of $\{a, c, d, f\}$ are f, j, h.

⑥ Lower bounds of $\{a, c, d, f\}$ is a.

The element x is the least upper bound of the subset A if x is an upper bound that is less than every other upper bound of A

Thus $a \leq x$ for any $a \in A$, and $x \leq z$ for any upper bound z of A. Thus a is the least upper bound of A.

If y is a lower bound of A and $z \leq y$ whenever z is a lower bound of A, then y is the greatest lower bound of A.

The greatest lower bound glb(A) and least upper bound lub(A) are unique - if they exist.

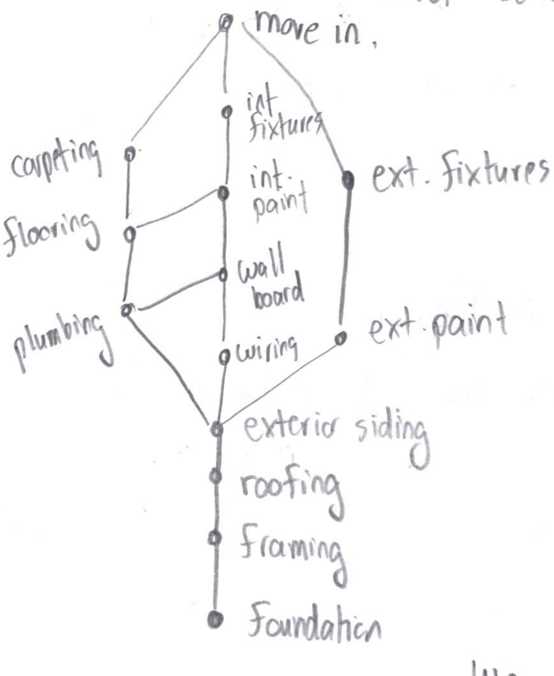
From our previous example:

lub of $\{b\ d\ g\}$ is 'g' (g & h are the upper bounds, g is the least of these)

glb of $\{b, b, g\}$ is 'b' (where a, b are the lower bounds and b is the greater of these).

Topological Sorting

Suppose we are building a house. We can define a partial order on the "must be done after" relation.



For building a house the steps we must take define a partial order and must be respected. But this doesn't specify a valid ordering - there could be many.

Eg: We don't care if we do the exterior painting or plumbing first.
- these are incomparable elements.

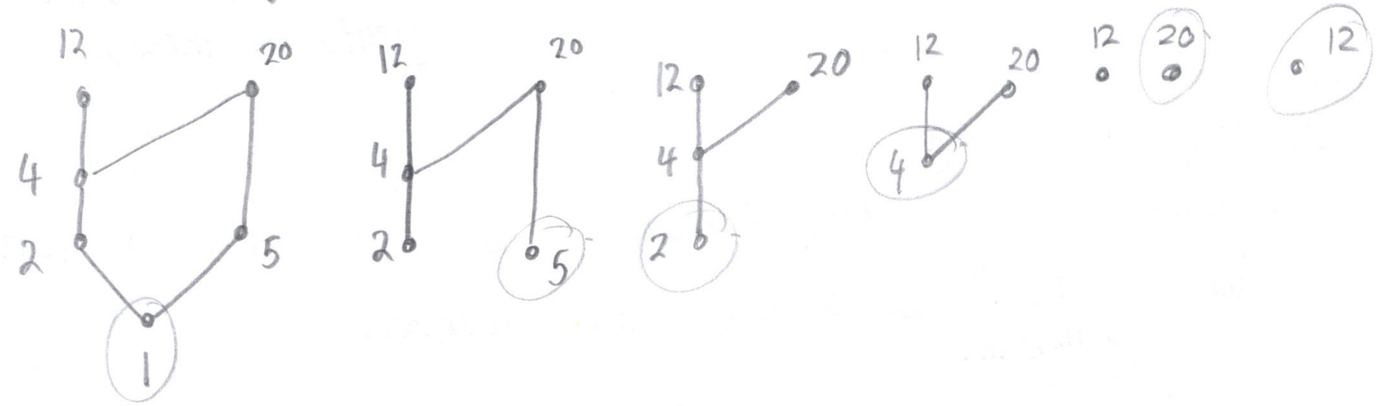
We want a total order that respects this partial order.

Def$^{\underline{n}}$: A total order $\preceq$ is compatible with a partial ordering R if $a \preceq b$ whenever aRb.

Observe that every finite, non-empty poset has at least one minimal element.

So to find a compatible total order, remove a minimal element and place it at the front of the total order. Now the initial partial order is a partial order with one fewer element. Keep doing this until all elements are gone. This is called <u>topological sorting</u>.

Ex:



(Topological sort of $(\{1,2,4,5,12,20\}, |)$

$$1 < 5 < 2 < 4 < 20 < 12$$

This is a valid total order given the '|' relation over the set in question.

END OF LECTURE