Lecture#2

Brief review : propositions & logic
- go over logical equivalences

Predicates & Quantifiers

In propositional logic how do you express the statement:

"Everyone in Toronto loves the Ottawa Senators"

- We cannot make this our proposition, too much info, we want simple statements with a single subject.

How can we determine, based on the statement above if the following is true:

t: "Bill loves the Ottawa Senators" (if Bill is from TO).

Idea: "loving the Ottawa Senators" is a property
"Everyone (in Toronto)" quantifies which people have this property. - Predicate Logic

Predicates

Consider the statement "X < 10", this is not a proposition since it depends on 'x'.

"person x loves the Ottawa Senators"

$$\underset{\substack{\uparrow \\ \text{subject} \\ \text{(variable)}}}{x} \; \underbrace{< \; 10}_{\text{predicate}}$$

Denote this: $P(x)$ ← value of a propositional function $\underline{P}$ at $\underline{x}$.

We can use this for any predicate
"> 3", "= 100", "is a cat", etc.

Once the propositional function is defined, we can use any object as the subject (variable value) to yield a truth value.

∴ if $P(x)$ = "$x < 10$" then

$P(5) = T$
$P(100) = F$
$P(10) = F$

We can also create more complex predicates with more than one variable

if $P(x,y)$ = "$x \geq y+1$" then
$P(5,2) = T$
$P(4,4) = F$

We are still faced with a problem. Our predicates still only solve half the problem
'x' is a person (from Toronto)

$L(x)$ = "person x loves the Ottawa Senators".

we would need to write

$L(x_1) \wedge L(x_2) \wedge L(x_3) \wedge \ldots$

we use QUANTIFIERS

QUANTIFIERS

We often wish to assert that a predicate is true for the elements of a particular domain - domain/univese of discourse.

For example the domain for our original statement is "Everyone in Toronto"

Consider $P(x)$ = " $x < 10$ "

Here domain may be all real numbers, ($\mathbb{R}$), all integers, all negative integers

"All People in Toronto" - wouldn't be suitable as a domain here!

The <u>universe of discourse</u> is the set of values you can plug in (use as variables) to your function: - can be <u>INFINITE</u>. (all integers/reals) -

Quantification expresses the extent to which a predicate is true over the <u>universe of discourse</u>. (<u>domain</u>)

UNIVERSAL QUANTIFICATION

For a statement $P(x)$ the <u>Universal Quantification</u> is the statement "$P(x)$ is true for all values of x in the domain".

Notation

$$\forall x P(x)$$

We can read this as "for all x, $P(x)$" or " for every x, $P(x)$".

For the domain $\{x_1, x_2 \ldots x_n\}$, $\forall x P(x)$ is shorthand for:

$$P(x_1) \land P(x_2) \land \ldots \land P(x_n)$$

- if for any element, say $\underline{P(x_2) = F}$, then $\forall x P(x)$ is also false, we call this a COUNTEREXAMPLE.
- it is sufficient to find <u>any</u> x_i for which $P(x_i) = F$ to prove that $\forall x P(x)$ is false.

Some examples (Universal Quantification).

Ex 1 $Q(x)$: "$x > 0$"

Domain: the set Z of Integers.

$\forall x Q(x)$ is false because $Q(-3)$ is false.

Domain: the set of positive integers.

$\forall x Q(x)$ is true.

The domain matters!

Ex 2:

$P(x)$: "$x^2 \geq x$"

Domain: What if we use $\mathbb{Z}$ (set of Integers)
$\mathbb{R}$ (set of Reals).

For $\mathbb{Z}$ - $x^2 \geq x$ is true for every integer $P(0)$ $(0^2 = 0)$
$P(1)$ $1^2 = 1$
$P(2)$ $2^2 > 2$
$P(-2)$ $(-2)^2 > -2$
etc.

For $\mathbb{R}$,
consider $(x = \frac{1}{2})$ $P(x) = \left(\frac{1}{2}\right)^2 \geq \frac{1}{2}$
$= \frac{1}{4} \not\geq \frac{1}{2}$ false.

So for the domain $\mathbb{R}$, $\forall x P(x)$ is false.

Existential Quantification

For $P(x)$ the Existential Quantification is the statement

"There exists an element x in the Domain s.t. $P(x)$ is true"

Notation

$$\exists x P(x)$$

Read as:

"There exists an x such that $P(x)$", "For some x, $P(x)$".

Domain $\{x_1, x_2 \ldots x_n\}$ - $\exists x P(x)$ is shorthand for $\Rightarrow$

$$P(x_1) \vee P(x_2) \vee P(x_3) \ldots \vee (x_n)$$ - disjunction.

If there is any $P(x_i) = T$ then $\exists x P(x)$ is true.

$$\forall x P(x) \rightarrow \exists x P(x)$$

Ex 1:

$P(x)$: "$x > 5$"

Domain: $\mathbb{Z}$

$\exists x P(x)$ is true because $P(10)$ "$10 > 5$" is true and $10 \in \mathbb{Z}$.

Ex 2: $P(x)$: "x is a TML fan."

Domain {"The set of people at Scotiabank Place"}

$\exists x P(x)$ is true if we find $P(\text{"Bill"})$ where Bill is at Scotiabank Place and Bill is a Toronto Maple Leaf fan.

Ex 3: $P(x)$: $x > x + 1$

Domain: $\mathbb{Z}$

$\exists x P(x)$ is false - there is no x s.t $x > x+1$.

Ex 4 $P(x)$: $x > 5$

Domain $\{2, 3, 5, 7, 9\}$

$\exists x P(x)$ is true because $P(7)$ is true.

Finding a single x for which $P(x)$ is true is sufficient to show $\exists x P(x)$ is true.

Precedence & Binding

$\forall, \exists$ have higher precedence than all logical operators.

eg. $\forall x\, P(x) \vee Q(x) \neq \boxed{\forall x (P(x) \vee Q(x))}$

is the disjunction of $\forall x P(x)$ and $Q(x)$ not

When we use a quantifier $\bar{w}$ a variable, we say that variable is <u>bound</u>.

- otherwise it is free - if not set to a particular value.

$$\exists x \underbrace{P(x)}_{\text{scope of } \exists x} \wedge Q(x)$$

- we cannot determine a value ($\therefore$ not a proposition)

but we can for

$\exists x P(x) \wedge Q(4)$ or $\leftarrow$ explict value for $Q(x)$

$\exists x P(x) \wedge \exists x Q(x)$ $\leftarrow$ both functions are bound.

also

$$\exists x \underbrace{(P(x) \vee Q(x))}_{\text{scope of this } 'x'} \wedge \forall x \underbrace{Z(x)}_{\text{scope of the second quantifier.}}$$

Note

$\exists x P(x) \wedge \exists x Q(x)$

these are not the same variable.

<u>Ex:</u> Domain: all people

$M(x)$: "x is a mathmatician."

$S(x)$: "x can sing"

$\exists x M(x) \wedge \exists x S(x) \rightarrow$ "at least one person is a mathmatician AND at least one person can sing"

$\exists x (M(x) \wedge S(x)) \rightarrow$ "~~at~~ least one person is a mathmatician who can sing"

Negating Quantifiers

Consider the quantified statement.

(1) "Every person likes math". (true if all people like math)

What if we wish to express the negation:

(2) "Not every person likes math". (true if there is any person who does not like math).

(2a) ↳ "there is a person who does not like math"

What about the following

(3) "At least one person likes math" ($\exists x P(x)$, $P(x)$: x likes math). (true if anyone likes math)

Negation

(4) "it is not the case that at least one person likes math".

or

(4a) "every person dislikes math"
(true only if all people in the domain dislike math.)

Notice that (From statements above)

(1) $\forall x P(x)$: $P(x)$: x likes math.

(2) $\neg \forall x\, P(x)$

(2a) $\exists x \neg P(x)$

these are logically equivalent.

$$\therefore \quad \boxed{\neg \forall x P(x) \equiv \exists x \neg P(x)}$$

(3) $\exists x\, M(x)$ $\qquad$ $M(x)$: x likes math.

(4) $\neg \exists x\, M(x)$

(4a) $\forall x \neg M(x)$

these are logically equivalent

$$\therefore \quad \boxed{\neg \exists x\, M(x) \equiv \forall x \neg M(x)}$$

De Morgans Laws For Quantifiers.

They are thus called because the quantifiers are shorthand for conjunction ($\forall x P(x) \equiv \{P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)\}$ and for disjunction ($\exists x P(x) \equiv \{P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)\}$

Remember those?

$$\neg(P_1 \vee P_2 \vee \ldots P_n) \equiv (\neg P_1 \wedge \neg P_2 \wedge \ldots \neg P_n) \quad \text{disjunction}$$

$$\neg(P_1 \wedge P_2 \wedge \ldots \wedge P_n) \equiv (\neg P_1 \vee \neg P_2 \vee \ldots \vee \neg P_n) \quad \text{conjunction.}$$

Example

Ex 1: What is the negation of $\forall x\ (x^2 > x)$?

$$\neg \forall x (x^2 > x) \equiv \exists x \neg (x^2 > x)$$
$$\equiv \exists x (x^2 \leq x)$$

Ex 2:

what is the negation of $\exists x (x^2 = 2)$?

$$\neg \exists x (x^2 = 2) \equiv \forall x \neg (x^2 = 2)$$
$$\equiv \forall x (x^2 \neq 2)$$

How to determine truth value of negated statements? (T2 p.41 text)

Negation	Equivalent To	Negation T when	Negation F when
$\neg \exists x P(x)$	$\forall x \neg P(x)$	for every x in domain $P(x) = F$	for some x in domain $P(x) = T$
$\neg \forall x P(x)$	$\exists x \neg P(x)$	there is an x in domain s.t. $P(x) = F$	for every x in domain $P(x) = T$

$\boxed{\text{10 minute break ?}}$

Translating English → Logical Expression ($\bar{w}$ predicates & quantifiers)

Universal Quantifier "$\forall x$"

- look for keywords "every" "all"

domain: all people

Ex 1: "Every student in this class will learn about logic"

$S(x)$ – x is a student in this class
$L(x)$ – x will learn about logic.

$$\Rightarrow \forall x \left(S(x) \rightarrow L(x) \right)$$

Why not?

$\forall x\, S(x) \rightarrow L(x)$ – no person is bound to $L(x)$

$\forall x \left(S(x) \wedge L(x) \right)$ – would only be true if the set of all people were students in this class.

Exesistential $\boxed{\text{"}\exists x\text{" look for "some" "there exists"}}$

Ex: "Some student in this class likes Miley Cyrus"

$S(x)$: "x is a student in this class"
$L(x)$: "x likes Miley Cyrus"

$$\Rightarrow \exists x \left(S(x) \wedge L(x) \right)$$

Why not?

$\exists x\, S(x) \wedge L(x)$ – no one bound to $L(x)$.

$\exists x \left(S(x) \rightarrow L(x) \right)$ – this is saying, there exists a person s.t. if they belong to this class, then they like Miley Cyrus – it does not say such a person exists – i.e empty class

<u>Ex:</u>
domain: all people in this class

"For every person y, if y likes hockey then y has attended a Senators game or y has attended an Ottawa 67's game."

$H(y)$: "y likes hockey."
$S(y)$: "y has attended a Senators game"
$O(y)$: "y " " " 67's game"

$$\forall y\,(H(y) \rightarrow (S(y) \vee O(y)))$$

Nested Quantifier chp. 1.4

Prop function may take more than one variable.
Let $F(x,y)$: "x and y are friends" domain: All people.

Then
$$\forall a \exists b\, F(a,b).$$

"Every person has at least one friend.
the order of the nesting matters, ie.

$$\exists b \forall a\, F(a,b)$$

"The exists a person who is friends with everyone"

<u>Example</u>: $M(x)$ "x is male"
$F(x)$ "x is female."
$L(x)$ "x is a student in this class"
$K(x,y)$ "x knows y".

What is: "Every female student in the class knows at least one male in the class"

$$\forall x\,((F(x) \wedge L(x)) \rightarrow \exists y\,(K(x,y) \wedge L(y) \wedge M(y)))$$

Ex: What is "No instructor knows everything"?

Domain: all people

$I(x)$: "x is an instructor"

$K(x)$: "x knows everything"

$$\neg \exists x \, (I(x) \wedge K(x)) \equiv \forall x \neg (I(x) \wedge K(x))$$

$$\equiv \forall x \, (\neg I(x) \vee \neg K(x)) \quad \text{- De Morgans.}$$

$$\equiv \forall x \, (I(x) \rightarrow \neg K(x)) \leftarrow \text{why? Imp. Equivalence}$$

Since $p \rightarrow q \equiv \neg p \vee q$ (Table 7, rule 1) so we have

$$\neg p \vee \neg q \equiv p \rightarrow \neg q.$$

Reads:

"If you are an instructor, then you don't know everything".

How would we express

"Some instructors don't know everything"?

$$\exists x \, (I(x) \wedge K(x))$$

Ex:

Domain : all athletes @ olympics.

$D(x)$: "x uses performance enhancing drugs"

$M(x)$: "x wins a medal".

How would we state the following expression in English.

$$\neg \forall x \, (\neg M(x) \rightarrow \neg D(x)) \; ?$$

Lets simplify first

$$\equiv \exists x \neg (\neg M(x) \rightarrow \neg D(x))$$

$$\equiv \exists x \neg (\neg\neg M(x) \vee \neg D(x)) \quad \boxed{\text{Table 7, rule 1}} \text{ I.E.}$$

$$\equiv \exists x \neg (M(x) \vee \neg D(x)) \quad \text{Double Negation}$$

$$\equiv \exists x \, (\neg M(x) \wedge \neg\neg D(x)) \quad \text{De Morgans Law Disjunction}$$

$$\equiv \exists x \, (\neg M(x) \wedge D(x)) \quad \text{Double Negation.}$$

Trans: "The exists an athlete who uses performance enhancing drugs and does not win a medal".

Ex: English "If a person is female and a parent, then they are someones mother"?

$F(x)$: "x is female"
$P(x)$: "x is a parent"
$M(x,y)$: "x is y's mother"

$$\forall x \left((F(x) \land P(x)) \longrightarrow \exists y \, M(x,y) \right)$$

Note: x is bound to all instances of 'x'

$$\equiv \forall x \exists y \left((F(x) \land P(x)) \to M(x,y) \right)$$

Ex: "Everyone has exactly one best friend."

$B(x,y)$: "y is x's best friend."

Domain: all people.

First consider "x has exactly one best friend".

$$(x), \exists y \left[B(x,y) \land \forall z \left((z \neq y) \to \neg B(x,z) \right) \right]$$

↑assume x is bound here

Now, over all x

$$\forall x \exists y \left[B(x,y) \land \forall z \left((z \neq y) \to \neg B(x,z) \right) \right]$$

Rules of Inference (1.5)

We should now know how to state things precisely, so we are ready to think about putting statements together to form arguments. A rigourous argument that is valid constitutes a proof. We need to put statements together using valid rules.

Ex ① "If it is cloudy, then it will rain"
② "It is cloudy".
} these statements are the premise

Conclusion: "It will rain"

This seems intuitively to be a correct (valid) argument.

An argument is valid if the truth of the premise imply the conclusion ie. premise $(P_1 \wedge P_2 \wedge \cdots \wedge P_m) \rightarrow C \equiv T$ then we have a valid argument.

A false premise may lead to a false conclusion.

We write arguments as:

"if it is cloud, then it will rain"
"it is cloudy"
∴ "it will rain"

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore q \end{array} \quad \text{(Modus Ponens)}$$

Likewise

"I am at home or I am at school"
"I am not at home"
"I am at school

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore q \end{array} \quad \text{(Disjunctive syllogism)}$$

Other Rules of Inference (Table 1 : p. 66 text)

$\dfrac{p}{\therefore p \vee q}$ Addition	$\dfrac{p \wedge q}{\therefore p}$ Simplification
$\dfrac{\begin{array}{c} p \\ q \end{array}}{\therefore p \wedge q}$ Conjunction	$\dfrac{\begin{array}{c} p \\ p \to q \end{array}}{\therefore q}$ Modus ponens.
$\dfrac{\begin{array}{c} \neg q \\ p \to q \end{array}}{\therefore \neg p}$ Modus tollens	$\dfrac{\begin{array}{c} p \to q \\ q \to r \end{array}}{\therefore p \to r}$ Hypothetical Syllogism
$\dfrac{\begin{array}{c} p \vee q \\ \neg p \end{array}}{\therefore q}$ Disjunctive Syllogism	$\dfrac{\begin{array}{c} p \vee q \\ \neg p \vee r \end{array}}{\therefore q \vee r}$ Resolution

To show that the premise imply the conclusion, we apply the rules of inference to the premise until we get the conclusion.

Ex:

"It is not sunny and it is colder than yesterday"

"We will go swimming only if it is sunny"

"If we don't go swimming then we will go biking"

"If we go biking we will be home by sunset".

Conclusion: "We will be home by sunset" (Valid?).

Let: y : "it is sunny"

c : "it is colder than yesterday"

s : "we will go swimming"

b : "we will go biking"

t : "we will be home by sunset"

1. $\neg y \wedge c$
2. $y \leftrightarrow s$ (so $s \to y$)
3. $\neg s \to b$
4. $b \to t$

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5. $\neg y$ — By Simplification of 1
6. $\neg s$ — Modus tollens on 2 and 5
7. b — Modus ponens 3 and 6
8. t — Modus ponens 4 and 7

$\therefore t$

And we have a valid argument.

You may replace premise by others which are logically equivalent.

Ex:

"If you lend me your game console I will play games."

"If you do not lend me your game console I will do my discrete math assignment"

"If I do my discrete math assignment I will be happy"

"Thus, if I do not play games I will be happy".

p: you lend me game console	r: I do my d.m. homework
q: I play games	h: I am happy

1. $p \to q$
2. $\neg p \to r$
3. $r \to h$ $\therefore (\neg q \to h)$

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4. $\neg q \to \neg p$ — Contrapositive of 1 - logically equivalent.
5. $\neg q \to r$ — Hypothetical Syllogism 2 and 4
6. $\neg q \to h$ — " " 3 and 5.

Ex

Either I study or I fail. ① $s \vee f$

I did not study. ② $\neg s$

∴ I fail! ∴ f Disjunctive Syllogism ① and ②.

Not all arguments are valid.

"If I do all the excercises I will get an A+. I got an A+, therefore, I did all the excercises."

$$\begin{array}{l} s \rightarrow a \\ a \\ \hline \therefore s \end{array}$$

To show invalidity find truth values to make all premise true, but the conclusion false. Recall to be valid

$$\underbrace{(P_1 \wedge P_2 \wedge \cdots P_m)}_{\text{all true}} \rightarrow \underbrace{C}_{\text{false}}$$

– then invalid.

So let $s = F$
$a = T$

then $s \rightarrow a$ is $F \rightarrow T \equiv T$

$a \equiv T$

but $s \equiv F$ so the argument is invalid.

<u>What about quantified statements?</u>

Until now we have reverted to propositional logic, for predicate logic, we need a few more rules of inference that allow us to deal with quantifiers.

Recall that $P(x)$ is a propositional function, so when x is an element of the domain it is simply a proposition that can be dealt with using rules of propositional logic.

Univesal Instantiation

$$\frac{\forall x\, P(x)}{\therefore P(c)}$$ - for any c in domain.

Existential Generalization

$$\frac{P(c)}{\therefore \exists x\, P(x)}$$ - for any c in domain

Univesal Generalization

$$\frac{P(c)}{\therefore \forall x\, P(x)}$$ - for an arbitrary c - ('any' is not sufficient)

Existential Instantiaton

$$\frac{\exists x\, P(x)}{\therefore P(c)}$$ - for some c.

We don't know which c is true, we only now that some 'c' is true.

Example

All men are mortal. Socrates is a man. Thus Socrates is mortal.

Domain: all things.

$M(x)$: "x is a man"
$D(x)$: "x is mortal"

① $\forall x\, (M(x) \rightarrow D(x))$
② $M(\text{Socrates})$ $\quad\therefore D(\text{Socrates})$

③ $M(\text{Socrates}) \rightarrow D(\text{Socrates})$ - Univesal Instantion ①.
④ $D(\text{Socrates})$ - Modus ponens ② and ③.

(up to p. 35 John's notes).

Ex: A student in this class has not read the book. Everyone in this class did well on the first assignment. Thus, someone who did well on the first assignment has not read the book.

$C(x)$ "x in class"
$B(x)$ "x read book"
$A(x)$ "x did well"

① $\exists x\,((C(x) \wedge \neg B(x))$
② $\forall x\,(C(x) \to A(x))$ $\quad \therefore \exists x\,(A(x) \wedge \neg B(x))$

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③ $C(a) \wedge \neg B(a)$ — Existential Instantiation ①.
④ $C(a)$ — Simplification ③.
⑤ $C(a) \to A(a)$ — Universal Instantiation ②
⑥ $A(a)$ — Modus ponens ④, ⑤
⑦ $\neg B(a)$ — Simplification ③
⑧ $C(a) \wedge A(a) \wedge \neg B(a)$ — Conjunction ⑥ and ⑦. and ④ *
⑨ $\exists x\,(C(x) \wedge A(x) \wedge \neg B(x))$ — Existential Generalization.

Ex: For every non-negative integer x, if $x > 1$, then $x - 1 > 0$.
All numbers x are > 1.
Thus, for all x, $x - 1 > 0$.

$P(x)$: $x > 1$ $\qquad$ Domain: $\mathbb{Z}^+ \setminus 0$
$Q(x)$: $x - 1 > 0$.

① $\forall x\,(P(x) \to Q(x))$
② $\forall x\; P(x)$ $\quad \therefore \forall x\, Q(x)$

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③ $P(c) \to Q(c)$ — Universal Instantiation ①.
④ $P(c)$ — " " ②
⑤ $Q(c)$ — Modus ponens ③ ④
⑥ $\forall x\, Q(x)$ — Universal Generalization ⑤ - since c is any arbitrary object from the domain.

Ex: If John knows discrete math, he will pass this course. John knows discrete math. Thus, everyone will pass the course. Domain: class.

1	$D(John) \rightarrow P(John)$	
2	$D(John)$	$\therefore \forall x\, P(x)$.
3	$P(John)$	Modus ponens 1 & 2
4	$\forall x\, P(x)$	Universal Generalization 3 ?

doesn't work, John is not an arbitrary object from domain, he is a specific instance.

$\boxed{\text{END OF LECTURE \#2}}$