LECTURE #4

Functions (2.3 In textbook)

Suppose we want to map one set to another as follows:

Given an element of set A (the Input)
return an element of set B (the Output).

Eg Set $A = \{x \mid x \text{ is a country}\}$
$B = \{x \mid x \text{ is a city}\}$

- We might want to know, given a country, what is that country's capital city.

  The input is the country, the output is the city.

- Another function might take as an input the name of a city, and return as its output the country that city is in.

Defn (Function)

Let A and B be two sets. A function from A to B is an assignment of exactly one element from B to each element of A.

write $f(a) = b$ if $b \in B$ is the unique element assigned by the function f to the element $a \in A$.

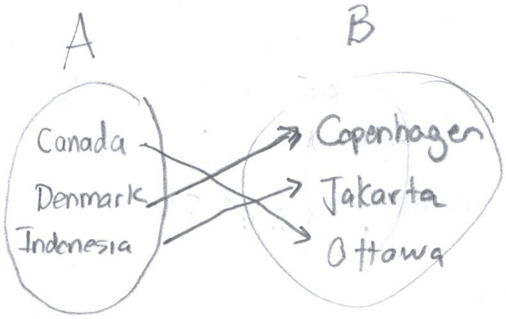
If f is a function from A to B write $f: A \to B$

Eg: $f: A \to B$ (A - countries, B - capital city)

'Canada'	'Ottawa'
'Denmark'	'Copenhagen'
'Indonesia'	'Jakarta'

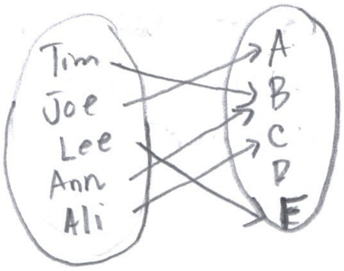
$f(\text{'Canada'}) = \text{'Ottawa'}$ $f(\text{'Denmark'}) = \text{'Copenhagen'}$

$f(\text{'Indonesia'}) = \text{'Jakarta'}$



The arrows are what defines the function 'p'.

In this example, each element has exactly one incoming arrow, this is not necessary. The uniqueness requirement is for the outgoing arrows.



– Assigning students a grade.

Functions can be specified in different ways:

- write out each pair : f('Canada') = 'Ottawa' ..
- By diagrams, as we have shown.
- A formula : $f(x) = 2x^2 + 1$
- By a program (or algorithm): input → output.

Recall we use $f : A \to B$ to denote formula f from A to B.

We say A is the domain of f

B is the codomain of f

f maps A to B

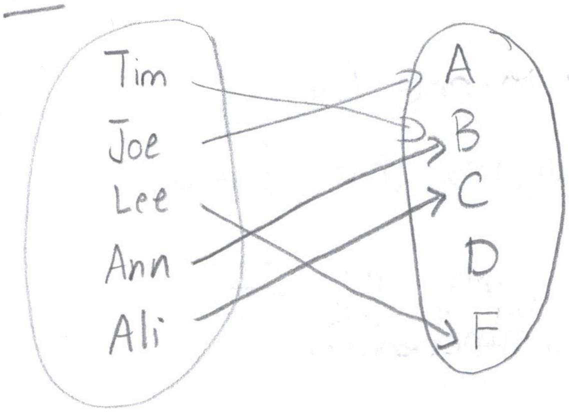
IF $f(a) = b$ then

'b' is the image of a

'a' is the preimage of b

The set of all images of elements of A is the range of f.

Ex



Domain. {Tim, Joe, Lee, Ann, Ali}
Codomain {A, B, C, D, F}
Range {A, B, C, F}

Since $f(\text{Ann}) = C$ then C is the image of Ann
Ann is the preimage of C.

Ex: Let $f: \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = x^2$

Domain: $\mathbb{Z}$
Codomain: $\mathbb{Z}$
Range $\{x, \mid x$ is a non-negative integer and a perfect square).
$= \{0, 1, 4, 9, 16, 25, \dots\}$

IF S is a subset of the domain we can also look at its image: the subset of B that consists of the images of the elements in S.

$$f(S) = \{f(s) \mid s \in S\}$$

Ex: From our grades example...

$$f(\{\text{Tim}, \text{Joe}, \text{Ann}\}) = \{A, B\}.$$

Notice in our grades example 'B' has one elements (Tim & Ann) mapped to it, while 'D' has none. We classify functions based on how they map elements from one set to the other.

One-to-One Function

A function f is said to be one-to-one (or injective) iff and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f.

- An injection is a one-to-one function.

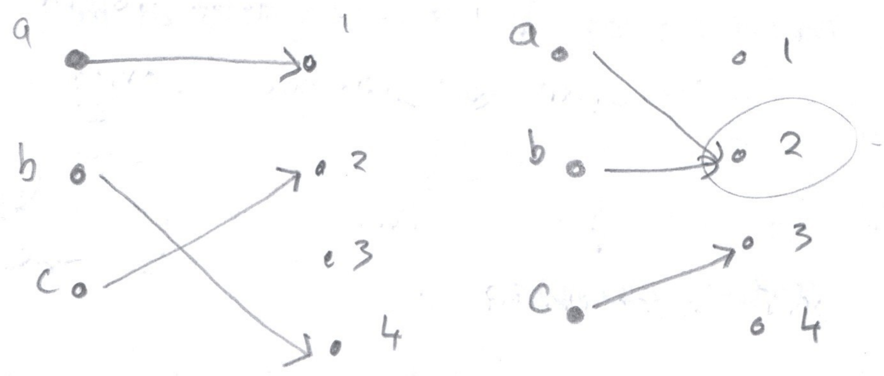
Recall by contraposition that:

$$((f(x) = f(y)) \rightarrow (x = y)) \text{ is equivalent to}$$

$$\neg(x = y) \rightarrow \neg(f(x) = f(y)) \equiv (x \neq y) \rightarrow (f(x) \neq f(y))$$

In a one-to-one function each element of the range has exactly one pre-image. Equivalently, each element of the codomain has at most one pre-image.

EX:



injective — not injective

To show a function is not injective give an x and y s.t. $x \neq y$ but $f(x) = f(y)$.

EX: Is $f(x) = x^2$ injective, $\mathbb{Z} \rightarrow \mathbb{Z}$.

Let $x = -1$ and $y = 1$, $x \neq y$ and $f(-1) = 1 = f(1)$

so this function is not injective.

Ex: Is $f(x) = 3x+2$ injective? $\mathbb{Z} \to \mathbb{Z}$

Assume $f(x) = f(y)$ then
$$3x + \cancel{2} = 3y + \cancel{2}$$
$$\cancel{3}x = \cancel{3}y$$
$$x = y \Rightarrow \text{Injective.}$$

Try this proof technique for $f(x) = x^2$

$f(x) = y(x)$ $\quad x^2 = y^2$
$$\sqrt{x^2} = \sqrt{y^2}$$
$\pm x = \pm y$, which is not the same as $x = y$! Not Injective.

Onto Functions

A function $f: A \to B$ is an onto (surjection) iff for every element $b \in B$, there is an element $a \in A$ with $f(a) = b$. Call such an f a surjection.

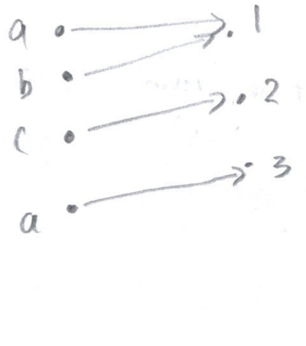
In an surjection every element of the codomain has a pre-image. Thus Equivalently the codomain and the range are the same. every element of B has one and only one incoming arrow.

To show a function is a surjection, start with an arbitrary element $b \in B$ and show that there is a preimage $a \in A$ such that $f(a) = b$.

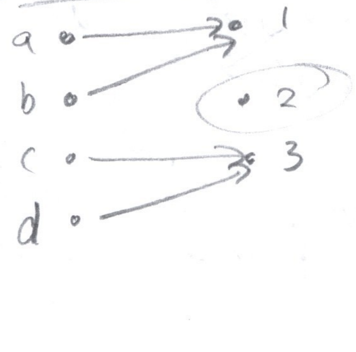
To show a function is not surjective give 'b' s.t $f(a) \neq b$ for any 'a'.

Ex:

Surjection



Not Surjection



$\neg \exists a (f(a) = 2)$

Ex: Is $P(x) = x^2$ surjective? $\mathbb{Z} \to \mathbb{Z}$

No: There is no x such that $x^2 = -1$ with x an integer.

Ex: Is $f(x) = 3x + 2$ surjective $\mathbb{R} \to \mathbb{R}$

Suppose we have an image $3x + 2 = y$, what pre-image does this yield.

$$3x = y - 2$$
$$x = \frac{y-2}{3}$$

when you do this, make sure this function is well def. over the codomain – it is in this case.

So to get an output of y, give input $\frac{y-2}{3}$ so the function is surjective.

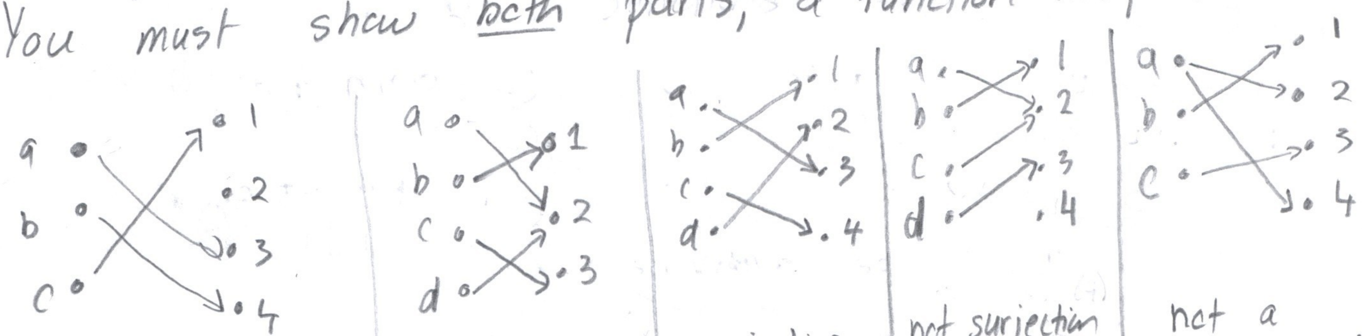
Note that: injective (one-to-one) $\iff$ at most one image.
surjective (onto) $\iff$ at least one image.

Defn
Bijection

A function that is both injective and surjective is called a bijection (or bijective).

To show a function is a bijection we must show:
1. it is an injection (see the above technique)
2. it is a surjection (" " " ")

You must show both parts, a function may be



injection (one to one)
Not (surjection, bijection)

surjection (onto)
NOT (injection, bijection)

surjection
injection
∴ bijection

not surjection
(no $f(x) = 4$)
not injection
($f(a) = 2 = f(b)$)
∴ not bijection

not a function!

The Invese of a Function

If f is bijective, then f is invertable. Its inverse is defined f^{-1} and assigns $b \in B$ to the unique element $a \in A$ s.t. $f(a) = b$.

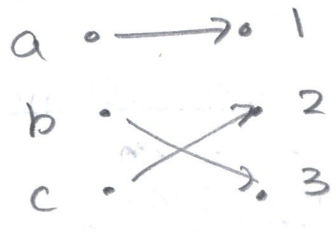
$$f^{-1}(b) = a \iff f(a) = b$$

Inversions are not defined for functions that are not bijections.

- If not injective then some $b \in B$ has two pre-images. Thus $f^{-1}(b) = ?$ - it has more than one value, so f^{-1} is not a function.
- If not surjective then there exists a $b \in B$ such that b has no pre-image, so $f^{-1}(b) = ?$ - it has no value so f^{-1} is not a function.

The inverse can be found by reversing the arrows in the diagram, or by isolating the other variable in the formula. Note the inverse of $f: A \to B$ is $f^{-1}: B \to A$

Ex:



- this in injective (each $a \in A$ only maps to a single element of B)
- this is surjective (there is an $a \in A$ mapped to each $b \in B$).

$f^{-1}(1) = a \quad f^{-1}(2) = c \quad f^{-1}(3) = b$.

Ex: $f(x) = 3x + 2$, $\mathbb{R} \to \mathbb{R}$

We already showed this was injective and surjective so it is bijective and invertable.

$$y = 3x + 2 \iff 3x = y - 2$$
$$x = \frac{y-2}{3}$$

$$\therefore f^{-1}(y) = \frac{y-2}{3} = x \text{ s.t } f(x) = y.$$

Countable & Uncountable Sets

A bijection can only exist between two sets if they have the same size. (cardinality) This allows us to reason about the sizes of infinite sets.

Consider $\mathbb{Z}^+ = \{1, 2, \ldots\}$; we call a set <u>countable</u> <u>iff</u>:

1. it is finite, or
2. it has the same cardinality as $\mathbb{Z}^+$

otherwise we say it is uncountable.

<u>Ex:</u> Are there more positive integers, or positive odd integers?

- This is the same as asking if the positive odd integers are countable
- This is the same as asking if there is a bijection from $\mathbb{Z}^+$ to $\{1, 3, 5, 7, 9, \ldots\}$

<u>Claim:</u> $f(n) = 2n - 1$ is a bijection mapping $\mathbb{Z}^+$ to the set of odd positive integers.

<u>Injective</u> (one-to-one): Suppose $f(n) = f(m)$, then

$$2n - 1 = 2m - 1$$
$$2n = 2m$$
$$n = m \quad \text{yes.}$$

<u>Surjective</u> (onto): Suppose $t \in \{1, 3, 5, 7, 9 \ldots\}$. Then let $t = 2n - 1$,

so $t = 2n - 1$,

$$t + 1 = 2n$$
$$\boxed{\frac{t+1}{2} = n}$$

So for any t, the value $k = \frac{t+1}{2}$ gives $f(k) = t$.

$\therefore$ $f(n)$ is both injective & surjective, so f is a bijection.

$\therefore$ there are equally many positive integers as there are positive odd integers!

What about the set of positive rational numbers. Countable?

Need: $f: \mathbb{Z}^+ \to \mathbb{Q}^+$

Note: we just need to <u>list</u> the positive rationals in some way, since the first element can be $f(1)$, the second $f(2)$, etc.

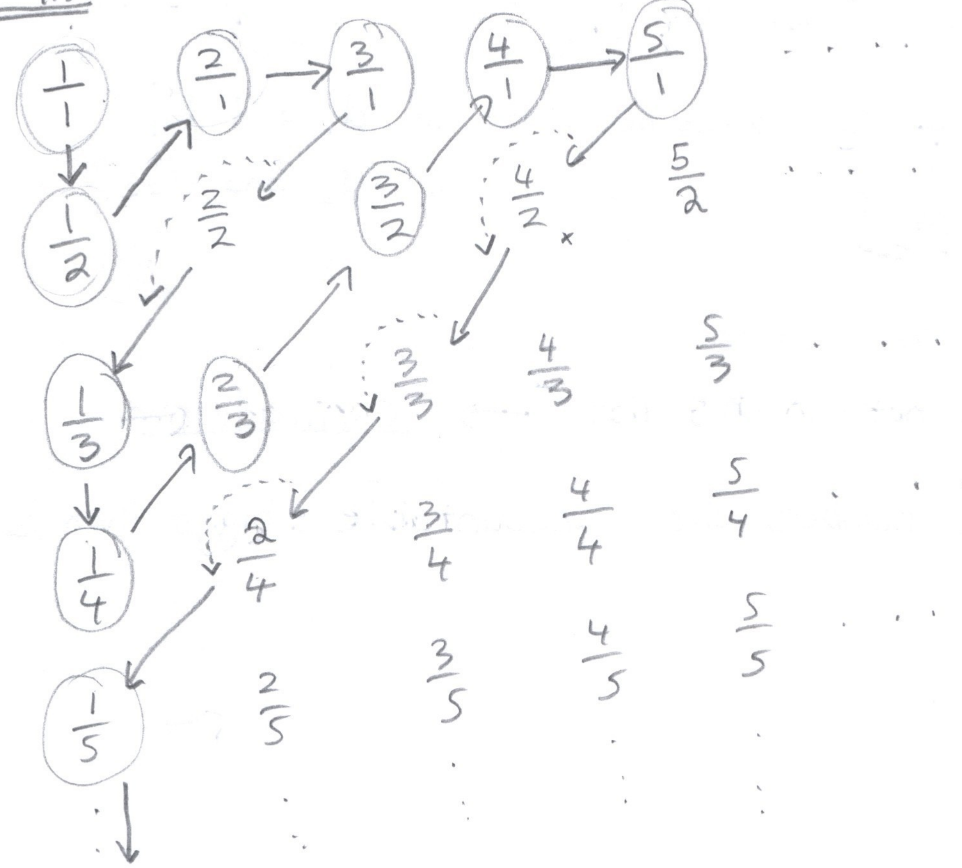
How for positive rational numbers?

A rational number is of the form $\frac{p}{q}$. Since we are dealing with positive rational numbers, we have $p, q \in \mathbb{Z}^+$.

Our list will consist of all rationals with $p+q=2$, then all with $p+q=3$, then $p+q=4$ and so on.

We do not write a number reducible to a number we have already written (eg. $\frac{2}{2} = \frac{1}{1}$). Note that there are only a finite number of rationals with a fixed k s.t $p+q=k$.

<u>Our list</u>



$\boxed{\text{10 Minute Break.}}$

Is $\mathbb{R}$ countable? (or $\mathbb{R}^+$).

Is $(0,1) \subseteq \mathbb{R}$ countable?

Supose it is : Thon we can list the elements:

$r_1 = 0.d_{11}\, d_{12}\, d_{13} d_{14} \ldots$

$r_2 = 0.d_{21}\, d_{22}\, d_{23} d_{24} \ldots$ where $d_{ij} \in \{0,1,..9\}$.

$r_3 = 0.d_{31}\, d_{32}\, d_{33} d_{44} \ldots$ (Exclude infite substrings of 9's).

$r_4 = 0.d_{41}\, d_{42}\, d_{43}\, d_{44}$

r will differ in circled digits.

Now, can we come up with a real number $\in (0,1) \in \mathbb{R}$ that is not on this list? Such a number will be a contradiction to our initial assumption.

Consider $r = 0.d_1 d_2 d_3 d_4 \ldots$ with

$$d_i = \begin{cases} 4 & \text{if } d_{ii} \neq 4 \\ 5 & \text{if } d_{ii} = 4 \end{cases}$$

So $d_i \neq d_{ii}$ for any i.

Now $r \neq r_1$ since they differ in d_1 and d_{11}

$r = r_2$ " " " " d_2 and d_{22}

and so on.

So 'r' is not in this list $\longrightarrow$ contradiction.

So, the real numbers are uncountable : bigger than $\mathbb{Z}^+$

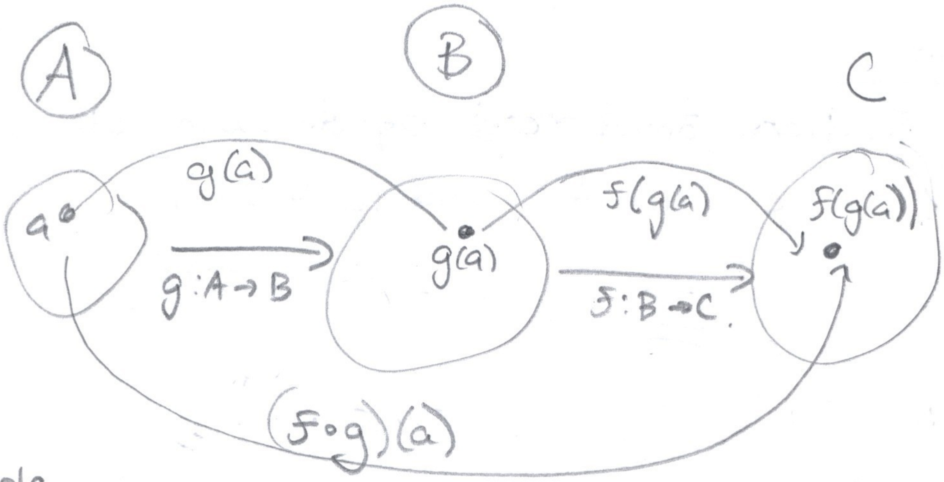
Composition of Functions.

Given two functions f and g we can use the output of one as the input of the other to create a new function:

$f(g(x))$: Evaluate g with input x.
Give result to f.
Overall output is this.

Defn: $f : B \to C$ and $g : A \to B$, then the composition of f and g is denoted $f \circ g$ (f follows g) and is defined by $(f \circ g)(x) = f(g(x))$

In order for $(f \circ g)$ to be defined, the range of g must be a subset of the domain of f.



Example

$g : \{a,b,c\} \to \{a,b,c\}$: $g(a) = b$ $g(b) = c$ $g(a) = a$
$f : \{a,b,c\} \to \{1,2,3\}$: $f(a) = 3$ $f(b) = 2$ $f(c) = 1$

What is $(f \circ g)(a)$

$(f \circ g)(a) = f(g(a)) = f(b) = 2$
$(b) = \quad = f(c) = 1$
$(c) \quad = f(a) = 3$

what is $(g \circ f)(a)$

$\underbrace{(g \circ f)(a) = g(f(a)) = g(3)\,?}$

not defined! The range of f is not a subset of the domain of g.

Ex: Let $f(x) = 4x - 2$, $g(x) = x + 7$

Then:

$$(f \circ g)(x) = f(g(x)) = f(x+7) = 4(x+7) - 2 = 4x + 26$$

$$(g \circ f) = g(f(x)) = g(4x-2) = (4x-2) + 7 = 4x + 5$$

In general $f \circ g \neq g \circ f$

One important case is composing a function with its inverse $f(a) = b$, then $f^{-1}(b) = a$.

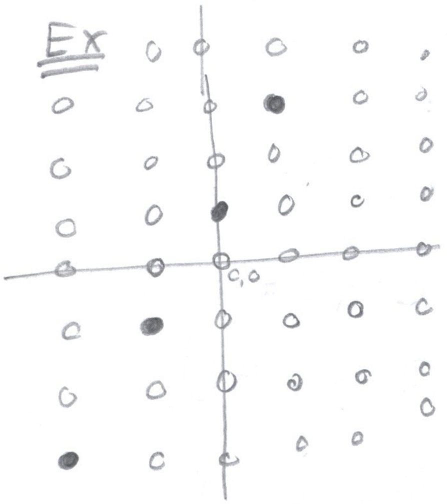
$$(f^{-1} \circ f)(a) = f^{-1}(f(a)) = f^{-1}(b) = a$$

$$(f \circ f^{-1})(b) = f(f^{-1}(b)) = f(a) = b$$

Graphs of Functions

We can plot a function $f : A \to B$ by drawing all points (a, b) where $a \in A$ and $b = f(a)$.

Ex

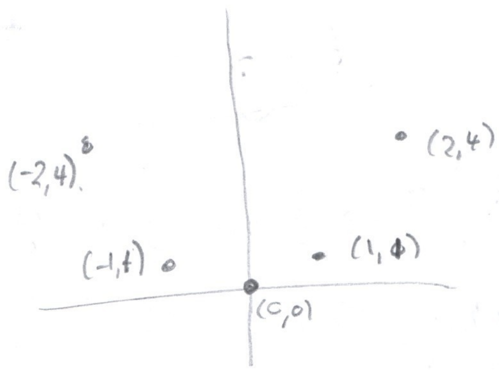


$f : \mathbb{Z} \to \mathbb{Z}$

$f(x) = 2x + 1$

o Hallow dots are all elements in cartesian product: $\mathbb{Z} \times \mathbb{Z}$.

Ex: $f(x) = x^2$ $\mathbb{Z} \to \mathbb{Z}$



$f : \mathbb{Z} \to \mathbb{Z}$

$f(x) = x^2$

Floors / Ceilings / Factorials

Floor : $\lfloor x \rfloor : \mathbb{R} \to \mathbb{Z}$: assigns largest integer $\leq x$.

Ceiling : $\lceil x \rceil : \mathbb{R} \to \mathbb{Z}$: assigns smallest integer $\geq x$.

If $x \in \mathbb{R}$ and $n \in \mathbb{Z}$ then

$$\lfloor x \rfloor = n \iff n \leq x \leq n+1$$
$$\lceil x \rceil = n \iff n-1 \leq x \leq n$$
$$\lfloor x \rfloor = n \iff x-1 < n \leq x$$
$$\lceil x \rceil = n \iff x \leq n < x+1$$

$$x-1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x+1$$

$$\lfloor -x \rfloor = -\lceil x \rceil$$
$$\lceil -x \rceil = -\lfloor x \rfloor$$

$$\left.\begin{aligned} \lfloor x+n \rfloor &= \lfloor x \rfloor + n \\ \lceil x+n \rceil &= \lceil x \rceil + n \end{aligned}\right\}$$ recall n is an integer.

$\lceil x+y \rceil = \lceil x \rceil + \lceil y \rceil$? Prove / Disprove. $x, y \in \mathbb{R}$.

$x = y = \frac{1}{2}$ $\quad \lceil x+y \rceil = \lceil \frac{1}{2}+\frac{1}{2} \rceil = \lceil 1 \rceil = 1$

$\lceil x \rceil + \lceil y \rceil = \lceil \frac{1}{2} \rceil + \lceil \frac{1}{2} \rceil = 1+1 = 2$. so this is not true.

Factorial: $f : \mathbb{N} \to \mathbb{N}$ $\quad f(n) = n!$

$$f(0) = 0! = 1$$
$$f(n) = n! = \overbrace{n \cdot (n-1) \cdot (n-2) \cdot \ldots 2 \cdot 1}^{n}$$

Ex: $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

SEQUENCES AND SUMS

A SEQUENCE is a function from a subset of $\mathbb{Z}$, usually $\{0,1,2,\ldots\}$ or $\{1,2,3,\ldots\}$ to a set S. We use a_n to denote the image of integer 'n'. We call a_n a term of the sequence. The sequence itself is denoted $\{a_n\}$.

Ex: $\{a_n\} = \frac{1}{n}$. The sequence $\{a_n\}$ is $a_1\ a_2\ a_3\ a_4 \ldots$

or $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

Geometric Sequence: a sequence of the form: (text uses Geometric Progression).

$$a, ar, ar^2, ar^3 \ldots ar^n$$

where a is the initial term $\in \mathbb{R}$.

r is the common ratio $\in \mathbb{R}$

Ex:

$\{b_n\}$ $b_n = 2 \cdot 5^n$ $a = 2$ $r = 5$ $2, 10, 50, 250, \ldots$

$\{c_n\}$ $c_n = \underset{\uparrow}{\ }(-1)^n$ $a = 1$ $r = -1$ $1, -1, 1, -1, 1, \ldots$

1.

$\{d_n\}$ $d_n = 6\left(\frac{1}{3}\right)^n$ $a = 6$ $r = \frac{1}{3}$ $6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \ldots$

An arithmetic sequence has the form:

$a, \quad a+d, \quad a+2d \quad \ldots \quad a+nd$

where a is the initial term
d is the common difference.

$\{s_n\} = 3 + 2n \quad a = 3 \quad d = 2 \quad \{3, 5, 7, 9, \ldots\}$

$\{t_n\} = -2 + 3n \quad a = -2 \quad d = 3 \quad \{-2, 1, 4, 7, \ldots\}$

Summations

One common operation on sequences is to compute the sum of certain portions of the sequence (or all).

Suppose we have

$a_1, \; a_2, \; a_3 \quad \ldots \quad a_m, \; a_{m+1}, \; a_{m+2} \; \ldots \quad a_n$

And we want to compute the sum from a_m to a_n.

$$a_m + a_{m+1} + a_{m+2} + \ldots + a_{n-1} + a_n$$

We can write this using Sigma notation:

upper limit $\rightarrow n$

$$\sum_{i=m}^{n} a_i$$

index of summation $\rightarrow i = m$ ← lower limit

pseudocode

```
S ← 0
for i = m to n do
    S ← S + a_i
return S.
```

Ex: The sum of the first 100 terms of $\{a_n\}$ where $a_n = \frac{1}{n}$ is

$$\sum_{i=1}^{100} a_i = \sum_{i=1}^{100} \frac{1}{i} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{100} \approx 5.2$$

Ex: Compute $\displaystyle\sum_{j=1}^{5} j^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2$

$= 1 + 4 + 9 + 16 + 25$

$= 55$

Sometimes we may want to change the upper or lower limits without changing the sum

Ex: $\sum_{j=1}^{5} j^2$ to be written with $j=0$ as lower limit and upper limit as 4. So we let $k=j-1$ and we sum.

$$\sum_{k=0}^{4} (k+1)^2$$

Also, we can split a sum up.

$$\sum_{i=1}^{n} a_n = \sum_{i=1}^{5} a_i + \sum_{i=6}^{n} a_i$$

This means that to exclude the first few terms we can say

$$\sum_{i=6}^{n} a_i = \sum_{i=1}^{n} a_i - \sum_{i=1}^{5} a_i$$

Summations can also be nested:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (i \cdot j)$$

Ex:

$$\sum_{i=1}^{4} \sum_{j=1}^{3} i \cdot j = \sum_{i=1}^{4} (i \cdot 1 + i \cdot 2 + i \cdot 3)$$

$$= \sum_{i=1}^{4} 6i$$

$$= 6 + 6 \cdot 2 + 6 \cdot 3 + 6 \cdot 4$$

$$= 6 + 12 + 18 + 24$$

$$= 60$$

When every term is multiplied by a constant term, we can factor it out :

$$\sum_{i=1}^{n} 6i = 6\sum_{i=1}^{n} i \qquad \text{so } 6\sum_{i=1}^{4} = 6\,(4+3+2+1)$$
$$= 6\,(10)$$
$$= 60$$

You can also split over addition.

$$\sum_{i=1}^{n} (i+2^i) = \sum_{i=1}^{n} i + \sum_{i=1}^{n} 2^i$$

This does not work for multiplication though. (with $n=4$)

$$40 = \sum_{i=1}^{n=4} i \cdot 4 \neq \underbrace{\sum_{i=1}^{n=4} i}_{10} \cdot \underbrace{\sum_{i=1}^{n=4} 4}_{16} = 160$$

One useful tool is the summation of a geometric sequence of the form ar^n

$$\sum_{j=0}^{n} ar^j = \begin{cases} \dfrac{ar^{n+1} - a}{r-1} & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1 \end{cases}$$

$$\boxed{\begin{array}{c} a, r \in \mathbb{R} \\ r \neq 0. \end{array}}$$

Why does this work?

Let $S = \sum_{j=0}^{n} ar^j$

Then

$$rS = r\sum_{j=0}^{n} ar^j$$

$$= \sum_{j=0}^{n} ar^{j+1} \quad \text{(we move the 'r' over)}$$

$$= \sum_{k=1}^{n+1} ar^k \quad \text{(we set } k = j+1 \text{, and } j = k-1)$$

$$= \sum_{k=0}^{n} ar^k + \left(ar^{n+1} - a\right)$$

the last term if we went to $n+1$

we start the summation at 0, so we add $k=0$, so we must remove the term $ar^0 = a$.

$$= S + \left(ar^{n+1} - a\right)$$

So:

$$rS = S + \left(ar^{n+1} - a\right)$$

$$rS - S = ar^{n+1} - a$$

$$S(r-1) = ar^{n+1} - a$$

For $r \neq 1$.

$$S = \frac{ar^{n+1} - a}{r-1}$$

If $r = 1$

because of the zero.

$$\sum_{j=0}^{n} a \cdot 1 = \sum_{j=0}^{n} a = (n+1)a$$

Some important summation formulas:

Know these.

$\sum_{k=0}^{n} ar^k \quad (r \neq 0)$	$\frac{ar^{n+1} - a}{r-1}, \; r \neq 1$	Geometric Sequence
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$	
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$	
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$	

these are called Closed Forms.

More Summation Formula (don't need to memorize).

$\sum_{k=0}^{\infty} x^k, \quad \lvert x \rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \quad \lvert x \rvert < 1$	$\frac{1}{(1-x)^2}$

closed forms.

How do we derive $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$?

$$S = \sum_{k=1}^{n} k$$

$$S = 1 + 2 + 3 + \ldots\ldots \; n-1 + n.$$

$$S = n + (n-1) + (n-2) + \ldots\ldots \; 2 + 1$$

$$2S = (n+1) + (n+1) + (n+1) + \ldots\ldots \; (n+1) + (n+1)$$

$$2S = n(n+1)$$

$$S = \frac{n(n+1)}{2}$$

$$\boxed{\text{END OF LECTURE \#4}}$$