# LECTURE # 8

Permutations & Combinations.

## Permutations

A permutation of a set of distinct objects is an ordered arrangement of these objects.

An ordered arrangement of $r$ elements of a set is called an $r$ - permutation.

Ex: $S = \{1,2,3,4\}$

$2,3,1,4$ is a permutation (or a 4 permutation)

$3,1,2$ is a 3-permutation

$4,2$ is a 2-permutation

The important thing is that order matters; so

$4,2$ and $2,4$ are different 2 permutations.

Many counting problems can be modelled as permutations. So how many $r$- permutations of a set of $n$ elements are there?

Denote an $r$-permutation of $P(n,r)$, we will use the Product rule to solve this:

$n$ ways to select the first element.

$n-1$ " " " " second "

$n-r+1$ " " " " $r^{th}$ element.

Total is $n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+1)$

$$= \frac{n!}{(n-r)!}$$

Why?

$$n! = \underbrace{n(n-1)(n-2)\cdots(n-r+1)}_{P(n,r)} \cdot \underbrace{(n-r)(n-r-1)\cdots 1}_{(n-r)!}$$

So $n! = P(n,r) \cdot (n-r)!$

thus
$$P(n,r) = \frac{n!}{(n-r)!}$$

Notice that $P(n,n) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} = n!$

Also $P(n,0) = \frac{n!}{(n-0)!} = \frac{n!}{n!} = 1$, since there is only one zero permutation – the empty set.

Ex: How many ways are there to award a first, second, and third prize from among 100 contestants in a race. (no ties!).

Note that in this case 'order matters' ("No one cares who was picked second")

#ways is a 3-permutation of a set of 100

$$P(100,3) = 100 \times 99 \times 98$$
$$= 970{,}200.$$

Ex: How many ways can you arrange a line of 10 students?

$P(10,10) = 10! = 3,628,800$

we are selecting from all 10.

Ex: How many permutations of the letters ABCDEFGH contain the string ABC.

$P(8,3)$? Doesn't work

We group 'ABC' as a single 'character' $\underbrace{\{ABC\}\ DEFGH}_{6 \text{ chars.}}$

$P(6,6) = 6! = 720.$

Combinations

An r-combination of a set is an unordered selection of r elements (= a subset of r elements).

Ex: $S = \{1,2,3,4\}$ then 3-combinations of S include

$\{4,1,3\}$
$\{2,4,1\}$
$\{1,2,3\}$
$\{3,4,1\}$

these two are the same.

Many counting problems can be modelled as combinations.
How many r-combinations are there of a set with $n$ elements?

$\Rightarrow$ Denote this by $C(n,r)$ or $\binom{n}{r}$ "$n$ choose $r$".

<u>Ex</u>: What is $C(4,2)$?

$\{a,b,c,d\}$: 2 combinations are all subsets of size 2.

$\{a,b\}$ $\{a,c\}$ $\{a,d\}$ $\{b,c\}$ $\{b,d\}$ $\{c,d\}$

$$C(4,2) = \binom{4}{2} = 6$$

What about a general rule?

Lets try writing the number of r-combinations in terms of the number of r-permutations.

The r-permutations of a set can be obtained by first forming the r-combinations, and then ordering the elements in each combination. This ordering may be done in $P(r,r)$ ways for each combination.

Thus $P(n,r) = C(n,r) \cdot P(r,r)$ so

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{\frac{n!}{(n-r)!}}{\frac{r!}{(r-r)!}} = \frac{n!\,(r-r)!}{(n-r)!\,r!} = \frac{n!}{r!\,(n-r)!}$$

Ex: How many subsets of size 6 can be selected from a set of size 10?

Choose 6 elements to be in the subset $C(10,6) = \binom{10}{6}$

Ex: How many ways to choose 5 people to be on a committee from a pool of 15 people.

- Since order doesn't matter : $\binom{15}{5}$

Ex: How many bit strings of length 'n' contain exactly r 1s.

$\binom{n}{r}$

Think of the problem this way if at first it doesn't make sense. Assume we have a bit string of length $n$ with all zero's, we can select any of $\binom{n}{r}$ subsets which we will set to '1'. These are exactly the set of bit strings with r 1's.

Notice then that this is also the number of ways which we can pick bitstrings of length $n$ that contain $n-r$ zeros. So.

$$\binom{n}{r} = \binom{n}{n-r}$$

$$\binom{n}{n-r} = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{(n-r)!\,r!} = \binom{n}{r}$$

Let's see some examples of counting using the various techniques we have seen so far:

Ex: Suppose we have an alphabet with the letters
$\{a, b, c, d, e, f, g, h\}$

How many

20 letter words : $\underbrace{8 \times 8 \times 8 \times 8 \ldots 8}_{20 \text{ times}} = 8^{20}$ by Product Rule.

12 letter words : $8 \times 8 \times 8 \times \ldots 8 = 8^{12}$ " " "

12 letter words (each letter distinct) : $0$ by Pigeonhole Princ.

5 letter words (distinct letters)? $P(8,5)$

14 letter words with exactly 3 'a's.

$\binom{14}{3} \cdot 7^{11}$

place the a's in one of 14 spots.

for the rest of the word we can choose from 7 letters (a total of 11 times).

12 letter words with $\leq 4$ b's

Cases : Exactly 4 : $\binom{12}{4} \cdot 7^{8}$

3 : $\binom{12}{3} \cdot 7^{9}$

2 : $\binom{12}{2} \cdot 7^{10}$

1 : $\binom{12}{1} \cdot 7^{11}$

0 : $\binom{12}{0} \cdot 7^{12}$

These are mutually exclusive so by the sum rule

$\binom{12}{4} \cdot 7^{8} + \binom{12}{3} \cdot 7^{9} + \binom{12}{2} \cdot 7^{10} + \binom{12}{1} \cdot 7^{11} + \binom{12}{0} \cdot 7^{11}$

14 letter words with <u>exactly</u> 2 'b's, 3 'a's and no 'd's.

$$\binom{14}{2}\binom{12}{3}5^9 \leftarrow \text{place the 9 non-}\{a,b,d\}\text{ letters.}$$

$\binom{14}{2}$ ↑ place 'b's

$\binom{12}{3}$ ↑ place 'a's

<u>10</u> letter words with <u>exactly</u> 4 'd's, which are consecutive.

a block of 4 'd's can appear as letters

1 ... 4
2 ... 5
3 ... 6
4 ... 7
5 ... 8
6 ... 9
7 ... 10

$\}$ 7 choices

Now fill in the remaining spaces 6 with any of the non 'd' letters. of which there are $7^6$

By product rule we have $7 \cdot 7^6 = 7^7$

<u>Equivalently</u> we could solve this as follows:

Let 'D' be a new character $\{dddd\}$.

Now we want to find out how many 7 letter words we can make from $\{a, b, c, D, e, f, g, h\}$ with just 1 D.

$$\binom{7}{1} \cdot 7^6 = 7 \cdot 7^6 = 7^7$$

$\binom{7}{1}$ ↑ to select the 'D'

$7^6$ ↑ the rest of the word has 6 characters and we can choose from among 7 options.