Lecture #9

Binomial Coefficients

The number of r-combinations of a n-set is denoted $\binom{n}{r}$. This symbol is called a binomial coefficient because they occur as coefficients in the expansion of binomial expressions such as $(a+b)^n$.

Ex: What is $(x+y)^n$?

$$(x+y)^3 = (x+y)(x+y)(x+y)$$
$$= (xx + xy + yx + yy)(x+y)$$
$$= (xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy)$$
$$= x^3 + 3x^2y + 3xy^2 + y^3$$

Think of all bit strings of length '3', the count 1s as x's, 0s as y's.

Observe that the terms x^3, x^2y, xy^2, y^3 are all possible combinations of x and y with 3 factors. Figuring out a coefficient means counting how many times a given combination appears.

Eg: The three ways we can choose 2 x's (and thus 1 y) are

1 2 3

$$(x+y)(x+y)(x+y)$$

So we have a coefficient of 3 on x^2y term.

To get x^3, x must be chosen in each of the 3 $(x+y)$ factors $\binom{3}{3}$

To get x^2y, x must be chosen in two (any y in the third $(x+y)$ factor $\binom{3}{2}$

To get xy^2, y must be chosen in two (and x in the third) $(x+y)$ factor $\binom{3}{2}$

To get y^2, y must be chosen in all three $(x+y)$ factors $\binom{3}{3}$

So we have $\binom{3}{3}x^3 + \binom{3}{2}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3$

$= x^3 + 3x^2y + 3xy^2 + y^3$

The Binomial Theorem

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j \qquad n \in \{0,1,2,3,\dots\}$$

Ex: Expand $(x+y)^4$

$$(x+y)^4 = \sum_{j=0}^{4} \binom{4}{j} x^{4-j} y^j$$

$$= \binom{4}{0}x^4y^0 + \binom{4}{1}x^3y^1 + \binom{4}{2}x^2y^2 + \binom{4}{3}x^1y^3 + \binom{4}{4}x^0y^0$$

$$= x^4 + 4x^3y^1 + 6x^2y^2 + 4xy^3 + y^4$$

Ex: What is the coefficient of the term $x^{12}y^{13}$ in the expansion of $(x+y)^{25}$?

A term looks like $\binom{25}{j}x^{25-j}y^j$, so $j = 13$ and we have $\binom{25}{13}x^{12}y^{13}$, so the coefficient of $x^{12}y^{13}$ is $\binom{25}{13} = 5{,}200{,}300$.

Ex: What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$?

$$(2x-3y)^{25} = ((2x)+(-3y))^{25}$$

$$\sum_{j=0}^{25} \binom{25}{j} 2x^{25-j}(-3y)^{j}$$

For $x^{12}y^{13}$ we have $j=13$ and thus

odd, so negative

$$\binom{25}{13} 2x^{12}(-3y)^{13} = \binom{25}{13} 2^{12} x^{12} (-3)^{13} y^{13}$$

$$\boxed{-\binom{25}{13} 2^{12} 3^{13}}$$ is the coefficient

Applications

Claim: Let n be a non-negative inter. Then

$$\sum_{k=0}^{n} \binom{n}{k} = 2^{n}$$

Proof:

Use Binomial theorem with $(1+1)^{n}$

$$\sum_{k=0}^{n} (1+1)^{n} = \binom{n}{0} 1^{n-0} 1^{0} + \binom{n}{1} 1^{n-1} 1^{1} + \cdots \binom{n}{n} 1^{0} 1^{n}$$

terms always 1 so

$$= \binom{n}{0} + \binom{n}{1} + \cdots \binom{n}{n}$$

$$= \sum_{k=0}^{n} \binom{n}{k}$$

$$= 2^{n}$$

This corollary of the Binomial Theorem is an alternate proof that the power set of a set on n elements is 2^n: by the sum rule consider the mutually exclusive cases of subsets having sizes $0, 1, 2 \ldots n$. There are $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots \binom{n}{n}$ such subsets, for a total of $\sum_{k=0}^{n} \binom{n}{k}$ subsets, which we have shown to be 2^n.

__Claim__: $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$

__Proof__: $0 = 0^n = (1+(-1))^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} (-1)^k$

always 1

$$= \sum_{k=0}^{n} \binom{n}{k} (-1)^k$$

This corollary implies that:

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots$$

Generalized Permutations & Combinations (Section 5.5)

Previously, when we discussed r-permutations of an n-element set, we did not allow for repetition. What if repetition is allowed?

By the Product Rule there are

$$\underbrace{n \cdot n \cdot n \cdot \ldots \cdot n}_{} = n^r \text{ "permutations"}$$

What about combinations with repetition?

Ex: How many ways to select 4 peices of fruit from a bowl containing apples, oranges, and pears, assuming that the bowl has 4 (or more) of each.

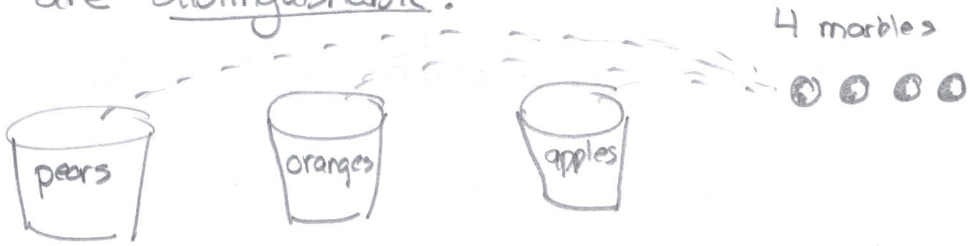
- Furthermore we don't care about order so which peice we pick doesn't matter, only the type of fruit.

4 apples	4 oranges	4 pears.
3 apples, 1 orange	3 oranges, 1 apple	3 pears, 1 apple
3 apples, 1 pear	3 oranges, 1 pear	3 pears, 1 orange
2 apples, 2 oranges	2 oranges, 2 pears	2 pears, 1 apple, 1 orange
2 apples, 1 orange & 1 pear	2 oranges, 1 apple & 1 pear	2 pears, 2 apples.

There are a total of 15 ways to pick 4 peices of fruit.

Doing such a listing would get tedious if we had say 100 peices of fruit to chose ... lets try and see how we can formalize this process.

Think of our possible selections as 'buckets' - these selections are <u>distinguishable</u>.

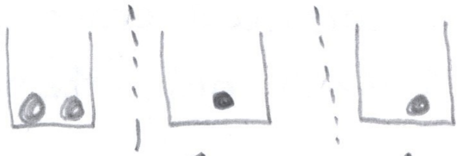


The the peices we pick are 4 <u>indistinguishable</u> marbles.

We take the 4 marbles and drop them one by one into one of the buckets.

The number of ways of picking the fruit = the number of ways of placing the marbles in the buckets.

eg.



2 pears 1 orange 1 apple.

Lets model this as follows let the marbles be represented by a '0' and the gaps between the buckets (dashed lines) by a '1'. Then we can represent any selection by a bit string.

"001010"

There are four 0 bits and two 1 bits (representing two divisions between the two buckets. How many six bit bitstrings are there with exactly two ones? $\binom{6}{2} = 15$.

In general, we want the number of r-combinations from a set of n-elements when repetition is allowed, we have

$n-1$ "1"s representing the divisions between types of elements.

r "0"s representing the number of picks we have.

So in total our bit string has length $n-1+r = n+r-1$ bits, r of which are "1"s, so we can count the number of r-combinations by

$$\binom{n+r-1}{r} = C(n+r-1, r)$$

Some examples

Ex: Suppose a pizza place has 4 types of pizza. How many ways can 6 pizza's be chosen (only the type of pizza matters - not the order).

→ 6 combinations of a 4 set (so 3 divisions) with repetition allowed

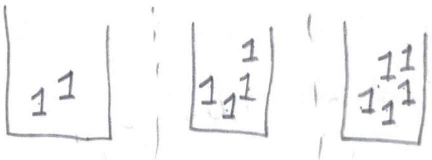
$$\text{"}\underbrace{010010100}_{6+4-1=9 \text{ (length)}}\text{"}$$

$$\binom{6+4-1}{6} = \binom{9}{6} = 84 = \binom{9}{3}$$

Ex: How many solutions to $x_1 + x_2 + x_3 = 11$ where x_1, x_2, and $x_3 \geq 0$ are integers. (Note if $x_1, x_2 x_3 \in \mathbb{Z}$ then there are infinite possibilities).

$$11 = 1+1+1+1+1+1+1+1+1+1+1$$

So we split the 11 '1's' needed to sum to 11 and place them in the 3 buckets x_1, x_2 and x_3



$x_1 = 2$ $\quad x_2 = 4$ $\quad x_3 = 5$ $\qquad 2+4+5 = 11$

So we are selecting 11 items from a set of 3 elements.

x_1 items of type 1
x_2 items of type 2
x_3 items of type 3

↳ 11- combinations with repetition from a 3-set.

$$\binom{3+11-1}{11} = \binom{13}{11} = 78.$$

r-permutations (order matters)	No repetition	$\frac{n!}{(n-r)!} = P(n,r)$
r-combinations (order doesn't matter)	No repetition	$\frac{n!}{r!(n-r)!} = C(n,r) = \binom{n}{r}$
r-permutations (order matters)	Repetition allowed.	n^r
r-combinations (order does not matter)	Repetition allowed	$\frac{(n+r-1)!}{r!(n-1)!} = C(n+r-1,r) =$

Permutations with Indistinguishable objects.

Sometimes the objects we are permuting are indistinguishable, in these cases we want to avoid double counting.

For example, consider the word:

FAILURE

There are $P(7,7) = 7! = 5040$ ways to arrange the letters of this word, but what about:

SUCCESS

Unless we label the individual letters many of the permutations will be indistinguishable. (ie. swap the double C's and S's and we still have the same word).

So how do we handle SUCCESS!

Try the following - we are placing 7 characters in 7 positions.

So first how many ways to place 3 S's . . . $\binom{7}{3}$

Second, ________ 2 C's $\binom{4}{2}$ - since 3 places are already accounted for.

We also have ________ 1 U $\binom{2}{1}$

And finally ________ 1 E $\binom{1}{1}$

$$\text{Total}: \frac{7!}{3!4!} \cdot \frac{4!}{2!2!} \cdot \frac{2!}{1!1!} = \frac{7!}{3!2!1!1!} = 420$$

In general this leads to a way to calculate the number of permutations with n_1 indistinguishable objects of type 1, n_2 of type 2, n_k of type k, etc.

$$\frac{n!}{n_1!n_2!\ldots n_k!}$$

Proof

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdot\cdots\cdot\binom{n-\sum_{i=1}^{k-1}n_i}{n_k}$$

$$= \frac{n!}{n_1!\cancel{(n-n_1)!}}\cdot\frac{\cancel{(n-n_1)!}}{n_2!\cancel{(n-n_1-n_2)!}}\cdot\frac{\cancel{(n-n_1-n_2)!}}{n_3!(n-n_1-n_2-n_3)!}\cdot\ \cdots$$

$$= \frac{n!}{n_1!n_2!\cdots n_k!}$$

Ex: How many different strings can be made by rearranging the letters of MISSISSIPPI

There are

$n_1 = 4$ I's

$n_2 = 4$ S's

$n_3 = 2$ P's

$n_4 = 1$ M

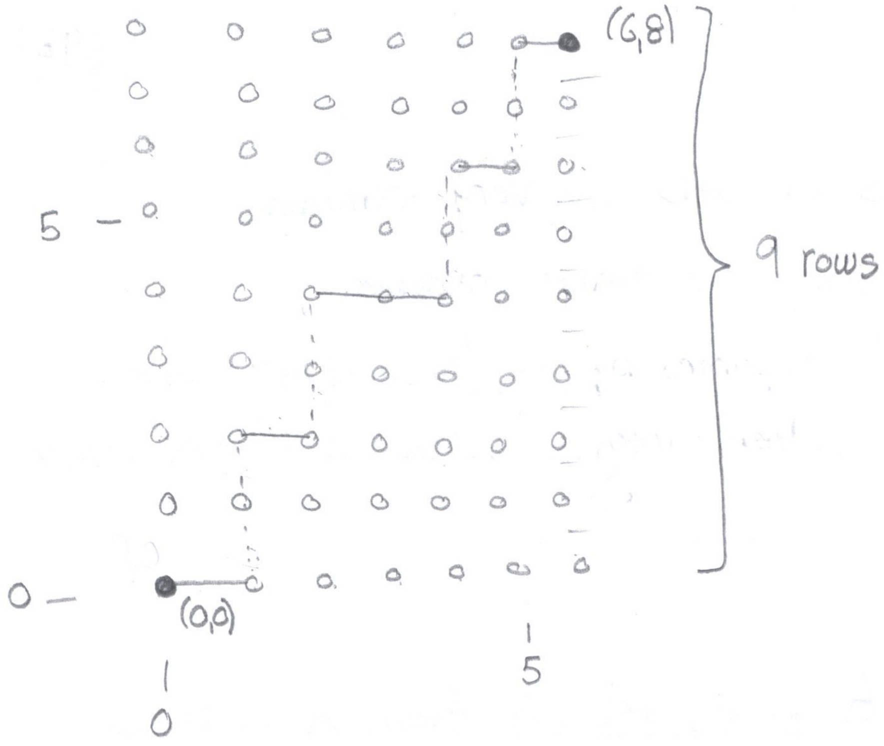
$\quad\;\; 11.$

$$= \frac{11!}{4!4!2!1!} = \frac{11\times \overset{5}{\cancel{10}}\times\cancel{9}\times\cancel{8}\times 7\times 6\times 5\times\cancel{4}\times 3\times 2\times 1}{\cancel{4}\times\cancel{4}\times\cancel{3}\times\cancel{3}\times\cancel{2}\times\cancel{2}\times 1\times 1} = \frac{11\times 5\times 7\times 6\times 5\times 3\times 2\times 1}{1}$$

$$= 69{,}300.$$

Ex: Suppose you are at $(0,0)$ on a grid, and want to go to $(6,8)$. How many paths are there if you can only move along the grid, up 1 $(y=y+1)$ or over 1 $(x=x+1)$ at any step.

Notice that six horizontal steps must be taken. If you specify where these are the vertical steps are already known.



Let the 9 rows represent 9 baskets into which we can place our horizontal lines.

This is the same as the placing marbles in baskets problem with apples, oranges, etc.

$$\binom{9+6-1}{6} = \binom{14}{6}$$

This is the same as determing where to place 6 1's in a bit string of length 15.

$$\underbrace{1001\ 0011\ 0010\ 01}$$

if we interpret bits as codes for 1 'OVER', 0 'UP' this is the path.

<u>Ex: 2</u>

How many solutions to

$$X_1 + X_2 + X_3 \leq 11$$

Rewrite

$$X_1 + X_2 + X_3 + X_4 = 11$$ and then solve $\binom{11+4-1}{4}$

$\binom{14}{4}$ solutions

In this case X_4 represents the number of unused values, or $11 - X_1 - X_2 - X_3$.

Relations

Relations between elements of sets are very common.

Eg: sets of people related by the father relation.

- employees related to companies by the 'employed by' relation..
- integers related to other integers by the 'divisible by' relation.

Formally, we say

Let A, B be sets. A <u>binary relation</u> from A to B is a subset of $A \times B$ (possibly empty).

So a binary relation is just a set of ordered pairs. We write aRb to mean $(a,b) \in R$, and $a \not R b$ to mean $(a,b) \notin R$.

When $(a,b) \in R$ we say "a is related to b by R".

We call these relations binary because $A \times B$ consists of pairs. In general, we can have relations as subsets of $A \times B \times C \times D \ldots$ to give us n-ary relations. We will concentrate on the binary case

<u>Ex:</u> Let A be the set of students at Carleton, and let B be the set of courses at Carleton. Let R be the "enrolled in" relation, where $(a,b) \in R$ (aRb) <u>iff</u> student a is enrolled in course b.

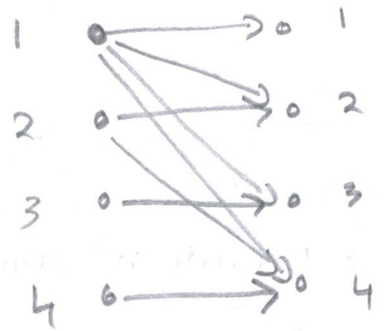
Note that all functions are relations, but not all relations are functions.

Sometimes a relation is between a set and itself. A subset of $A \times A$ is an example. In this case we say the relation is on the set A.

__Ex__: Let $A = \{1,2,3,4\}$, what ordered pairs are in the relation $R = \{(a,b) \mid a \text{ divides } b\}$?

> From Sec 3.4 (which we haven't covered).
> If $a,b \in \mathbb{Z}$ with $a \neq 0$, we say a divides b if there is an integer c such that $b = ac$.

$$R = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3)(4,4)\}$$



$\overbrace{\qquad\qquad}^{b}$

R	1	2	3	4
a: 1	X	X	X	X
2		X		X
3			X	
4				X

Notice many outgoing edges, which is impossible with functions.

Relations on $\mathbb{Z}$

Ex: $R_1 = \{(a,b) \mid a \le b\}$

$R_2 = \{(a,b) \mid a > b\}$

$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$

$R_4 = \{(a,b) \mid a = b\}$

$R_5 = \{(a,b) \mid a = b + 1\}$

$R_6 = \{(a,b) \mid a + b \le 3\}$

$(1,1) \in R_1 R_2 R_3 R_6$

$(1,2) \in R_1 R_6$

$(2,1) \in R_2 R_5 R_6$

$(1,-1) \in R_2 R_3 R_6$

$(2,2) \in R_1 R_3 R_4$

How many relations are there on a set of n elements?

Equivalent to, how many subsets of $A \times A$?

$$|A \times A| = n^2 \quad (\text{if } |A| = n)$$

$$\therefore \quad |P(A \times A)| = 2^{n^2}$$

– since a set with m elements has 2^m subsets, a set with n^2 elements has 2^{n^2}.

Ex: How many relations are there on the set $\{a, b, c\}$.

$$2^{n^2} = 2^{3^2} = 2^9 = 512$$

Properties of Relations

Reflexivity

A relation R on a set A is reflexive if $(a,a) \in R$ for all $a \in A$.

Ex: If $A = \{1,2,3\}$ then

$R_1 = \{(1,1), (1,2), (2,2), (2,3), (3,3)\ (3,1)\}$

is reflexive while

$R_2 = \{(1,1), (1,2), (2,3), (2,1), (3,2)\}$ is not.

$R_3 = \{(1,2)\ (1,3)\ (2,3)\ (3,1)\}$ is not.

Ex: How many reflexive relations are there on a set of n elements.

A relation on a set A is a subset of $|A \times A|$ (n^2 pairs).

A set with m elements has 2^m subsets. (where a relation is determined by specifying which of the n^2 pairs is in R).

We know the n pairs (where $a = a$ for $a \in A$) are already in R - so don't consider these.

Therefore each of the remaining $n \cdot n-1$ ordered pairs of the form (a,b) with $(a \neq b)$ may, or may not, be in R. Thus this set has $2^{(n \cdot n-1)}$ subsets.

Ex: Is the 'divides' relation reflexive?

Yes: a divides a since $a = 1 \cdot a$ for any integer a.

Symmetry & Antisymmetry

A relation R on a set A is called <u>symmetric</u> if $((a,b) \in R \to (b,a) \in R))$ for all $a,b \in A$.

A <u>relation</u> R on a set A is called <u>antisymmetric</u> if $((a,b) \in R \wedge (b,a) \in R) \to (a=b)$ for all $a,b \in A$.

In other words if $(a \neq b)$ then either $(a,b) \notin R$ or $(b,a) \notin R$.

Note that these properties are not mutually exclusive, a relation could have one, both, or neither.

<u>Ex</u>: Consider the following relations on $\{1,2,3,4\}$

$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$

↳ not symmetric $(3,4)$ but no $(4,3)$.
→ not antisymmetric $(1,2)$ and $(2,1)$ and $1 \neq 2$.

$R_2 = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)\}$

↳ symmetric
↳ not antisymmetric $(1,2)$ and $(2,1)$, $(1,4)$ and $(4,1)$.

$R_3 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)$

↳ not symmetric $(2,1)$ not $(1,2)$.
↳ antisymmetric

$R_4 = \{(1,1), (2,2), (3,3)\}$

- symmetric
- antisymmetric

$\boxed{\text{END OF LECTURE.}}$