This is a closed book test. No calculators are allowed. All questions should be answered on this sheet in the space provided. If you need you need addtional space you may use the back of the page (but be sure to indicate where your answer is). You have 1 hour and 30 minutes to complete the exam. There are a total of 34 marks over 6 pages. Please write you name and student number on EACH page.

1 Logical Equivalances

Name	Equivalence
Identity laws	$p \wedge \mathbf{T} \equiv$ $p \vee \mathbf{F} \equiv$
Domination laws	$p \vee \mathbf{T} \equiv$ $p \wedge \mathbf{F} \equiv$
Idempotent laws	$p \vee p \equiv$ $p \wedge p \equiv$
Double Negation law	$\neg(\neg p) \equiv$
Commutative laws	$p \vee q \equiv$ $p \wedge q \equiv$
Associative laws	$(p \vee q) \vee r \equiv$ $(p \wedge q) \wedge r \equiv$
Distributive laws	$p \vee (q \wedge r) \equiv$ $p \wedge (q \vee r) \equiv$
De Morgan's laws	$\neg(p \wedge q) \equiv$ $\neg(p \vee q) \equiv$
Absorption laws	$p \vee (p \wedge q) \equiv$ $p \wedge (p \vee q) \equiv$
Negation laws	$p \vee \neg p \equiv$ $p \wedge \neg p \equiv$
Implication Equivalence	$p \rightarrow q \equiv$
Biconditional Equivalance	$p \leftrightarrow q \equiv$

2 Rules of Inference

Name	Equivalence		
Modus Ponens	p $\underline{p \rightarrow q}$ $\therefore$	Modus tollens	$\neg q$ $\underline{p \rightarrow q}$ $\therefore$
Hypothetical syllogism	$p \rightarrow q$ $\underline{q \rightarrow r}$ $\therefore$	Disjunctive syllogism	$p \vee q$ $\underline{\neg p}$ $\therefore$
Addition	$\underline{p}$ $\therefore$	Simplification	$\underline{p \wedge q}$ $\therefore$
Conjuction	p $\underline{q}$ $\therefore$	Resolution	$p \vee q$ $\underline{\neg p \vee r}$ $\therefore$

3 Rules of Inference for Quantified Statements

Universal instantiation	$\underline{\forall x P(x)}$ $\therefore$	Universal generalization	$\underline{P(c) \text{ for an arbitrary } c}$ $\therefore$
Existential instantiation	$\underline{\exists x P(x)}$ $\therefore$	Existentail generalization	$\underline{P(c) \text{ for some element } c}$ $\therefore$

Multipe Choice Questions

1. (**2 marks**) Let $A = \{\{\emptyset\}, c, d, \{a, b, e\}, c, f\}$ what is the cardinality of the powerset of A?

   (a) 16
   (b) 6
   (c) 32
   (d) 64

   Solution:
   b. There are 5 elements in the set $\{\emptyset\}$, c, d, and $\{a, b, e\}$ so there are $2^5 = 32$ elements in the power set.

2. (**2 marks**) Consider the function $f : A \rightarrow B$ that is one-to-one but not onto on the finite sets A and B, which of the following statements is true?

   (a) $|A| > |B|$
   (b) $|A| \geq |B|$
   (c) $|A| < |B|$
   (d) $|A| \leq |B|$

   Solution:
   If $|A| > |B|$ the function cannot be *one-to-one* so that rules out a and b. If f is *one-to-one* and $|A| = |B|$ then f would be an *onto* function, which rules out d. Thus $|A| < |B|$ so the correct answer is c.

3. (**2 marks**) The statement "'Not every student in this class will learn about logic"' is equivalent to which of the following statements in predicate logic, where $S(x)$ is "'x is a student in this class"' and $L(x)$ is "'x learns about logic"'.

   (a) $\forall x(S(x) \wedge \neg L(x))$
   (b) $\exists x(\neg S(x) \vee \neg L(x))$
   (c) $\exists x(S(x) \wedge \neg L(x))$
   (d) $\exists x(S(x) \wedge L(x))$

   Solution:
   c is the correct answer. The original statement is equivalent to $\neg\forall x(S(x) \rightarrow L(x))$ so we have:

$$
\begin{aligned}
\neg\forall x(S(x) \rightarrow L(x)) &\equiv \exists x \neg(S(x) \rightarrow L(x)) && \text{Neg. Quant} \\
&\equiv \exists x \neg(\neg S(x) \vee L(x)) && \text{Implication Equivalence} \\
&\equiv \exists x(\neg\neg S(x) \wedge \neg L(x)) && \text{De Morgans} \\
&\equiv \exists x(S(x) \wedge \neg L(x)) && \text{Double negation}
\end{aligned}
$$

   This statment is equivalent to "'There exists a person who is a student in this class that does not learn about logic"'.

Long Answer Questions

4. (**2 marks**) Is the logical statement $(p \leftrightarrow q) \vee (p \oplus q)$ a tautology, a contingency, or a contradiction.

   Solution:
   This statement is a tautology. Since $(p \leftrightarrow q)$ is true whenever p and q have the same truth value, while $(p \oplus q)$ is true whenever p and q have differing truth values, taking the logical or of these two statements is always true for any p and q. This could be shown using a truth table, using an argument like that presented here, or for the brave at heart using logical equivalences.

   Grading:
   2 marks. 1/2 mark for the correct answer, 1 1/2 marks for good argument/truth table.

5. (**1 mark**) How many rows would be in the truth table for the following compound proposition: $(p \vee q) \wedge \neg(q \wedge t) \vee (r \rightarrow s)$

   Solution:
   2^5 or 32

   Grading:

6. (**2 marks**) Indicate which of the following conditional statements are true, and which are false.

   (a) If $2 + 1 = 4$ then $2 + 2 = 5$.

   (b) If $1 + 1 = 2$ then $2 + 1 = 4$.

   Solution:

   (a) T since hypothesis is F.
   (b) F since hypothesis is T and conclusion is F.

   Grading:
   1 mark for each.

7. (**2 marks**) Use logical equivalences to determine whether the statements $(p \rightarrow r) \wedge (q \rightarrow r)$ and $(p \vee q) \rightarrow r$ are logically equivalent (you cannot use truth tables). As was done class and the assignments you should justify the steps you are taking by naming the logical equivalence you are using.

   Solution:
   We begin by simplifying the first statement.

$$
\begin{aligned}
(p \rightarrow r) \wedge (q \rightarrow r) &\equiv (\neg p \vee r) \wedge (\neg q \vee r) && \text{Impl.Equiv x 2} \\
&\equiv (r \vee \neg p) \wedge (r \vee \neg q) && \text{Commutative} \\
&\equiv r \vee (\neg p \wedge \neg q) && \text{Distributive}
\end{aligned}
$$

   And transforming the second we have:

$$
\begin{aligned}
(p \vee q) \rightarrow r &\equiv \neg(p \vee q) \vee r && \text{Impl.Equiv} \\
&\equiv (\neg p \wedge \neg q) \vee r && \text{De Morgans} \\
&\equiv r \vee (\neg p \wedge \neg q) && \text{Commutative}
\end{aligned}
$$

   So the statements are logically equivalent.

   Grading:
   Full marks for showing the two statements are logically equivalent. The demonstration of the commutative equivalence is included for completeness, but if students use it implicity that is OK they can still get full marks. Take off 1/2 mark if the student fails to indicate which equivalences they are using but is otherwise correct.

8. (**2 marks**) Express the following statement using mathematical and logical operators, predicates and quantifiers. The statement is:

   For all integers the product of a positive and a negative integer is a negative integer.

   Solution:
   Let $P(x)$ be x is positive, and let $N(x)$ be x is negative. Then we can express this as follows:

   $\forall_x \forall_y ((P(x) \wedge N(y)) \rightarrow N(x \cdot y))$

   or

   $\forall_x \forall_y (((x > 0) \wedge (y > 0)) \rightarrow (x \cdot y > 0))$

   Grading:

9. (**3 marks**) Use rules of interference and quantifiers to determine if the following argument is valid, where the universe of discourse (domain) is all people. "Some students in the class did not read the textbook", "Every student in the class passed the test." Therefore, someone who passed the test did not read the textbook.

Solution:

First we list our propositions.

- Let C(x) denote "x is in the class".
- Let P(x) denote "x passed the test".
- Let T(x) denote "x read the textbook".

In the arguments that follow *s* is used to represent a specified element of the domain (a student). We want the conclusion $\exists x(P(x) \land \neg T(x))$.

Step	Reason
1. $\exists x(C(x) \land \neg T(x))$	Premise
2. $\forall x(C(x) \to T(x))$	Premise
3. $C(s) \land \neg T(s)$	Ex. Instantiation from 1
4. $C(s)$	Simplification from 3.
5. $C(s) \to P(s)$	Universal Instantiation from 2.
6. $P(s)$	Modus Ponens 4 and 5.
7. $\neg T(s)$	Simplificaiton from 3.
8. $P(s) \land \neg T(s)$	Conjuction from 6 and 7.
9. $\exists x(P(x) \land \neg T(x))$	Existential Generalization 8.

Grading:

10. (**3 marks**) Calculate and simplify the value of the summation $\sum_{k=x}^{y} c \cdot k$ in terms of x, y and c.

Solution:

$$\begin{aligned}
\sum_{k=x}^{y} c \cdot k = c\sum_{k=x}^{y} k &= c\sum_{k=1}^{y} k - c\sum_{k=1}^{x-1} k \\
&= c\left(\frac{y(y+1)}{2}\right) - c\left(\frac{(x-1)x}{2}\right) \\
&= \frac{c}{2} \cdot (y^2 + y) - \frac{c}{2} \cdot (x^2 - x) \\
&= \frac{c}{2} \cdot (y^2 + y - x^2 + x)
\end{aligned}$$

Grading:

11. (**2 marks**) Draw the Venn diagram for the following combination of the sets A, B, and C: $(A \cap \overline{B}) \cup (A \cap \overline{C})$.

Solution:

Grading:

12. (**2 marks**) Calculate the closed form of the summation $\sum_{j=0}^{12} (2)3^j$

Solution:

This is a geometric sequence with $a = 2$ and $r = 3$ so we have:

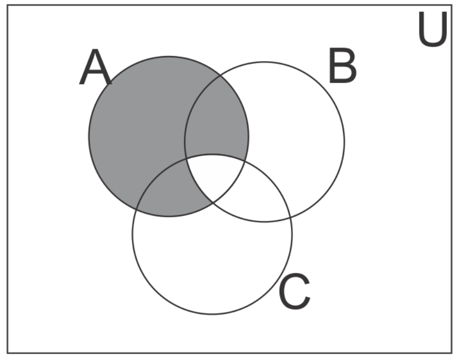

Figure 1: Solution to Venn diagram question.

$\sum_{j=0}^{n} ar^j = \frac{ar^{n+1} - a}{r-1}$
$= \frac{2 \cdot 3^{12+1} - 2}{3-1}$
$= \frac{2 \cdot 3^{13} - 2}{2}$
$= \frac{2(3^{13} - 1)}{2}$
$= 3^{13} - 1$

Grading:

13. (**2 marks**) Let n be an integer. Prove that n is even if and only if $5n + 6$ is even.

    Solution:
    First we prove n is even implies $5n + 6$ is even. In this case $n = 2k$ so $5n + 6 = 5(2k) + 6 = 10k + 6 = 2(5k + 3)$. If we let $t = 5k + 3$ then we have this equal to $2t$ so $5n + 6$ is even.

    Now we want to show $5n + 6$ is even implies n is even. We do this by contraposition. Try to prove $O(n) \rightarrow O(5n+6)$ where $O(x)$ means x is odd. So assume n is odd $n = 2k+1$ then $5n+6 = 5(2k+1)+6 = 10k+5+6 = 10k + 10 + 1 = 2(5k + 5) + 1$ if we let $t = 5k + 5$ then we have this is equal to $2t + 1$, which is and odd number.

    Grading:
    Give most of the marks if the students gets that they need to prove both ways even if one of the proofs is incorrect. Give full marks if both proofs are correct.

14. (**2 marks**) Give a proof by contradition that the sum of a rational number and an irrational number is irrational.

    Solution:
    To prove this by contradiction we assume that the sum of a rational number and and irrational number is rational and show this leads to a contradiction.

    So assume the sum of an irrational number z and a rational number $\frac{a}{b}$ is a rational number $\frac{c}{d}$. Then $z + \frac{a}{b} = \frac{c}{d}$. Thus $z = \frac{c}{d} - \frac{a}{b} = \frac{cb-ad}{bd}$, so z must be a rational number, which is a contradiction.

    Grading:

15. (**3 marks**) Noting that $3^6 = 729$ and $6! = 720$ prove that $3^n < n!$ when n is an integer greater than 6 using mathematical induction.

    Solution:
    BASIS STEP: Let $n = 7$ (since n must be greater than six) so we have $3^7 = 2187$ and $7! = 7 \cdot 720 = 5040$, so our claim is true for the basis step.

    INDUCTIVE HYPOTHESIS: Assume $3^k < k!$ when $k > 6$.

    INDUCTIVE STEP: We wish to prove that $3^{k+1} < (k + 1)!$. We can do so as follows:

$$\begin{aligned}(k+1)! &= (k+1)\cdot k! \\ &> 3\cdot k! && \text{Since } k>6. \\ &> 3\cdot 3^k && \text{By the inductive hypothesis.} \\ &= 3^{(k+1)}\end{aligned}$$

Thus $3^{k+1} < (k+1)!$. ∎

Grading:
If a student just lists the steps and nothing else, give 1 mark. Then give an extra 1/2 mark for each part they get right and the extra 1/2 mark for getting everything right. If a student is very close on a number of steps (but not quite right) they should get 1 1/2 for the whole question.

16. (**2 marks**) Is $f(n) = n^3 - 2n^2 + 156n + 10974$ is $f(n) \in O(n^5)$? Is $f(n) \in O(n^2)$? Justify your answers.

Solution:
$f(n)$ is $O(n^5)$ since $n^3 - 2n^2 + 156n + 10974 < n^5 + 156n^5 + 10974n^5 < 11131n^5$ so we can choose as witnesses $c = 11131$ and $k = 1$.

$f(n)$ is not $O(n^2)$. Try to pick a suitable c such that $c \cdot n^2 > n^3 - 2n^2 + 156n + 10974$.

$$\begin{aligned}c\cdot n^2 &> n^3 - 2n^2 + 156n + 10974 \\ c &> n - 2 + \frac{156}{n} + \frac{10974}{n^2}\end{aligned}$$

We just have to pick $n = c + 1$ for this to be false.

Grading: