Chapter 1

1. Suppose that a program solves the following problem.
   input : An array $A[1..n]$ of n numbers.
   output : Two indices i and j such that $1 \leq i < j \leq n$ and the difference $A[j] - A[i]$ is maximised.

   What output should the program return if we provide it the input $[100, 0, -1]$?
2. Suppose that a program solves the following problem.
   input : An array $A[1..n]$ of n numbers.
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   What output should the program return if we provide it the input $[100, 0, -1]$?
3. Suppose that a program solves the following problem.
   input : An array $A[1..n]$ of n numbers.
   output : Two indices i and j such that $1 \leq i \leq n$, $1 \leq j \leq n$ and the difference $A[j] - A[i]$ is maximised.

   What output should the program return if we provide it the input $[100, 0, -1]$?
4. Suppose that a program solves the following problem.
   input : An array $A[1..n]$ of n numbers.
   output : Two indices i and j such that $1 \leq i \leq n$, $1 \leq j \leq n$, $i \neq j$ and the difference $A[j] - A[i]$ is maximised.

   What output should the program return if we provide it the input $[100, 0, -1]$?
5. Respond to the questions 1 to 4 by changing "maximised" by "minimized".

—— Inference rules ——

6. Is the following reasoning valid ? Explain why. Let

   P : If George does not have eight legs, then he is not a spider.
   Q : George is a spider.

$$\frac{\begin{array}{c} P \\ Q \end{array}}{\text{George has eight legs.}}$$

7. Do the hypotheses "If it does not rain or if there is no fog, then the sailing race will take place and the trophy will be awarded," and "the trophy was not awarded," imply the conclusion "it rained" ? Explain why.
8. Write each of the following arguments using inference rule notation. State if the reasoning is valid and explain why.
   (a) Daniel, a student in this class, knows how to write programs in Java. Anyone who can write programs in Java can get well-paid work. Therefore, a person in this class can get a well-paying job.
   (b) Someone in this class enjoys watching whales. Anyone who enjoys whale watching cares about ocean pollution. Therefore, there is someone in this class who cares about ocean pollution.

—— Proofs ——

For each of the following statements,

— Write the statement by using quantifiers and logical operators.

— Determine if the statement is true or false.

— Show that your response is correct.

— Indicate the method of proof used.

9. Each odd integer corresponds to the difference of two squares.
10. (a) Each even integer corresponds to the difference of two squares.
    (b) Each even integer corresponds to the difference of two different squares.
11. If n is an integer and $n^3 + 5$ is odd, then n is even.
12. At least ten out of any 64 days of the year 2020 must land on the same day of the week.
13. There exists a calendar year with thirteen "Friday the 13ths".
14. Let x be an integer. If $3x + 2$ is even, then $x + 5$ is odd.
15. Let x be an integer. If $x + 5$ is odd, then x^2 is even.
16. Let x be an integer. If x^2 is even, then $3x + 2$ is even.
17. The number $\sqrt{5}$ is irrationnal.
18. The number $\sqrt[3]{2}$ is irrationnal.
19. The number $\log_2(3)$ is irrationnal.
20. If $a > 0$ and b are two rational numbers, then a^b is a rational number.

21. If n is a positive integer, we have

$$\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

22. If n is a positive integer, we have

$$\sum_{i=0}^{n} (2i+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}.$$

23. If n is a positive integer, we have

$$\sum_{i=0}^{n} 3 \cdot 5^i = \frac{3(5^{n+1}-1)}{4}.$$

24. If n is an integer greater than 3, we have $2^n > n^2$.
25. If n is a positive integer, then $n^3 - n$ is divisible by 3.
26. If n is a positive integer, we have

$$\sum_{i=0}^{n} i \cdot 2^i = (n-1)2^{n+1} + 2.$$

— Two final questions —

27. Let $P(n)$ be a propositional formula. Determine for which positive integers n the proposition $P(n)$ must be true in each of the following cases.
    (a) $P(1)$ is true and, for all positive integers n, $P(n) \to P(n+2)$.
    (b) $(P(1) \wedge P(2))$ is true and, for all positive integers n, $(P(n) \wedge P(n+1)) \to P(n+2)$.
    (c) $P(1)$ is true and, for all positive integers n, $P(n) \to P(2n)$.
    (d) $P(1)$ is true and, for all positive integers n, $P(n) \to P(n+1)$.
28. What is the problem with this "proof" that all horses are the same colour ?

    Let $P(n)$ = "all horses of any group of n horses are the same colour."

    Base step : Clearly, $P(1)$ is true.

    Induction hypothesis : Let $k \geq 1$ and suppose that $P(k)$ is true. Otherwise stated, all horses of any group of k horses are the same colour.

Induction step : Consider $k+1$ horses : enumerate the horses $1, 2, 3, ..., k, k+1$. If we consider only the first k horses, we therefore have a group of k horses. By the induction hypothesis, they are all the same colour. If we consider only the last k horses, we therefore have a group of k horses. By the induction hypothesis, they are also all the same colour. But because the first k horses and the k last horses have horses in common, then the $k+1$ horses are the same colour !