Next, we need a particular solution for

$$a_n = 5a_{n-1} - 6a_{n-2} + 7^n$$

Since $F(n) = 7^n$ is exponential, let's try a solution of the form

$$a_n^{(P)} = c\,7^n$$

We get

$$c7^n = 5\cdot c7^{n-1} - 6c7^{n-2} + 7^n$$
$$49c7^{n-2} = 35\cdot c\cdot 7^{n-2} - 6c7^{n-2} + 49\cdot 7^{n-2}$$
$$49c = 35\cdot c - 6c + 49$$
$$c = \frac{49}{20}$$

So $a_n^{(P)} = \frac{49}{20}\cdot 7^n$

So $a_n = a_n^{(P)} + a_n^{(h)}$

$$= \frac{49}{20}\cdot 7^n + \alpha_1\cdot 2^n + \alpha_2\cdot 3^n$$

$$0 = a_0 = \frac{49}{20} \cdot 7^0 + \alpha_1 \cdot 2^0 + \alpha_2 \cdot 3^0$$

$$= \alpha_1 + \alpha_2 + \frac{49}{20}$$

$$0 = a_1 = \frac{49}{20} \cdot 7^1 + \alpha_1 \cdot 2^1 + \alpha_2 \cdot 3^1$$

$$= 2\alpha_1 + 3\alpha_2 + \frac{343}{20}$$

So $\alpha_1 = \frac{49}{5}$

$\alpha_2 = \frac{-49}{4}$

$$a_n = \frac{49}{20} \cdot 7^n + \frac{49}{5} \cdot 2^n - \frac{49}{4} \cdot 3^n$$

Rather than "guess" the form of the particular solution, we can use the following theorem.

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Course 20

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Theorem

Let

$$a_n = c_1\, a_{n-1} + c_2\, a_{n-2} + \dots + c_k\, a_{n-k} + F(n),$$

where $c_1, c_2, \dots, c_k \in \mathbb{R}$ *with* $c_k \neq 0$ *and*

$$F(n) = \left(b_t\, n^t + b_{t-1}\, n^{t-1} + \ldots + b_1\, n + b_0\right) s^n,$$

where $b_0, b_1, \dots, b_t, s \in \mathbb{R}$ *with* $b_t \neq 0$ *and* $s \neq 0$.

- *Assume that s is not a root of the characteristic equation of the associated linear homogeneous recurrence relation. Then there is a particular solution $a_n^{(p)}$ of the form*

$$\left(p_t\, n^t + p_{t-1}\, n^{t-1} + \ldots + p_1\, n + p_0\right) s^n.$$

- *Otherwise, s is a root of the characteristic equation of the associated linear homogeneous recurrence relation. Assume that s is a root with multiplicity m. Then there is a particular solution $a_n^{(p)}$ of the form*

$$n^m \left(p_t\, n^t + p_{t-1}\, n^{t-1} + \ldots + p_1\, n + p_0\right) s^n.$$

Example

$$a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3} + (n^2+1)2^n$$
$$a_0 = 0$$
$$a_1 = 1$$
$$a_2 = 2$$

This recurrence is linear and non-homogeneous of order $k = 3$. To solve it, we need the solution $a_n^{(h)}$ to the associated linear homogeneous recurrence relation :

$$a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3}$$

We know now how to proceed. We find

$$a_n^{(h)} = \alpha_1 + \alpha_2 \cdot 2^n + \alpha_3 \cdot n \cdot 2^n,$$

where α_1, α_2 and α_3 are constants.

To find a particular solution $a_n^{(p)}$, we use the previous theorem. To apply this theorem, we need the roots of the associated linear homogeneous recurrence relation :

$$r^3 - 5r^2 + 8r - 4 = (r-1)(r-2)^2$$

Here, $s = 2$ is a root with multiplicity $m = 2$. Hence, by the theorem, there is a particular solution $a_n^{(p)}$ of the form

$$a_n^{(p)} = n^2 \cdot (p_2 \cdot n^2 + p_1 \cdot n + p_0)\, 2^n,$$

for some constants p_0, p_1 and p_2.

We find

$$
\begin{aligned}
n^2(p_2 n^2 + p_1 n + p_0)2^n = \ & 5(n-1)^2\Big(p_2(n-1)^2 + p_1(n-1) + p_0\Big)2^{n-1} \\
& - 8(n-2)^2\Big(p_2(n-2)^2 + p_1(n-2) + p_0\Big)2^{n-2} \\
& + 4(n-3)^2\Big(p_2(n-3)^2 + p_1(n-3) + p_0\Big)2^{n-3} \\
& + (n^2+1)2^n
\end{aligned}
$$

We divide by 2^n on both sides of the equality. After simplification, we find

$$
p_2 n^4 + p_1 n^3 + p_0 n^2 = p_2 n^4 + p_1 n^3 + (-6p_2 + p_0 + 1)n^2 - 3p_1 n + (11p_2 - p_0 + 1)
$$

Hence,

$$
\begin{aligned}
p_0 &= -6p_2 + p_0 + 1 \\
0 &= -3p_1 \\
0 &= 11p_2 - p_0 + 1
\end{aligned}
$$

Then we find

$$p_0 = \frac{17}{6} \qquad p_1 = 0 \qquad p_2 = \frac{1}{6}$$

Hence,

$$a_n^{(p)} = n^2 \left(\frac{1}{6} n^2 + \frac{17}{6} \right) 2^n$$

from which

$$\begin{aligned} a_n &= a_n^{(p)} + a_n^{(h)} \\ &= n^2 \left(\frac{1}{6} n^2 + \frac{17}{6} \right) 2^n + \alpha_1 + \alpha_2 \cdot 2^n + \alpha_3 \cdot n \cdot 2^n \end{aligned}$$

It remains to find the values of α_1, α_2 and α_3 to satisfy the initial conditions.

We have

$$0 = a_0 = 0^2 \cdot \left(\frac{1}{6} \cdot 0^2 + \frac{17}{6}\right) 2^0 + \alpha_1 + \alpha_2 \cdot 2^0 + \alpha_3 \cdot 0 \cdot 2^0 = \alpha_1 + \alpha_2$$

$$1 = a_1 = 1^2 \cdot \left(\frac{1}{6} \cdot 1^2 + \frac{17}{6}\right) 2^1 + \alpha_1 + \alpha_2 \cdot 2^1 + \alpha_3 \cdot 1 \cdot 2^1 = 6 + \alpha_1 + 2\alpha_2 + 2\alpha_3$$

$$2 = a_2 = 2^2 \cdot \left(\frac{1}{6} \cdot 2^2 + \frac{17}{6}\right) 2^2 + \alpha_1 + \alpha_2 \cdot 2^2 + \alpha_3 \cdot 1 \cdot 2^2 = 56 + \alpha_1 + 4\alpha_2 + 8\alpha_3$$

Thus,

$$\alpha_1 = -34 \qquad \alpha_2 = 34 \qquad \alpha_3 = -\frac{39}{2}$$

Therefore the final solution is

$$\begin{aligned} a_n &= n^2 \left(\frac{1}{6} n^2 + \frac{17}{6}\right) 2^n - 34 + 34 \cdot 2^n - \frac{39}{2} \cdot n \cdot 2^n \\ &= \frac{1}{6} \left(n^4 + 17n^2 - 117n + 204\right) 2^n - 34 \end{aligned}$$