CSI-2101 Discrete Structures (Summer 2020)
~~Mini-Test~~ Assignment #4

- DEADLINE: Thursday, July 16, 2020 at noon
- **The deadline is strict. There are only two questions. No late assignment will be tolerated.**
- TO BE SUBMITTED ON BRIGHTSPACE. You must submit one single pdf file to BrightSpace.

1. (10 marks) For this question, you will need your student number.

   (a) (0 mark) What is your student number?

   Let t be the last digit of your student number. Consider the following non-homogeneous linear recurrence (where you have to replace t by the last digit of your student number):

$$\begin{aligned} a_n &= -a_{n-1} + 6a_{n-2} + 125(t+1)\cdot(n+1)\cdot 2^n \\ a_0 &= 0 \\ a_1 &= 0 \end{aligned}$$

   (b) (2 marks) Find the solution $a_n^{(h)}$ to the associated homogeneous linear recurrence.

   SOLUTION: The associated homogeneous linear recurrence is

$$a_n = -a_{n-1} + 6a_{n-2}.$$

   Thus, the characteristic equation is $x^2 + x - 6 = (x-2)(x+3)$. Thus, from the theorem seen in class, we have

$$a_n^{(h)} = \alpha_1 \cdot 2^n + \alpha_2 \cdot (-3)^n,$$

   for some constants α_1 and α_2.

   (c) (6 marks) Find a particular solution $a_n^{(p)}$ to the non-homogeneous linear recurrence. I strongly suggest that you use the theorem from Course 19.

   SOLUTION: In the following solution, replace t by the last digit of your student number and you will find your personnalized solution!
   Since 2 is a root with multiplicity $m = 1$ of the characteristic equation, from a theorem seen in class, there is a particular solution of the form

$$a_n^{(p)} = n^1\left(p_1 n^1 + p_0\right)2^n = n\left(p_1 n + p_0\right)2^n,$$

   for some constants p_0 and p_1.
   We substitute in the recurrence to find the values of p_0 and p_1. The recurrence

$$a_n = -a_{n-1} + 6a_{n-2} + 125(t+1)\cdot(n+1)\cdot 2^n$$

becomes

$$n\Big(p_1n+p_0\Big)\, 2^n = -(n-1)\Big(p_1(n-1)+p_0\Big)\, 2^{n-1}+6(n-2)\Big(p_1(n-2)+p_0\Big)\, 2^{n-2}+125(t+1)\cdot(n+1)\cdot 2^n.$$

We divide by 2^{n-2} on both sides of the equality and we find

$$4n\Big(p_1n+p_0\Big) = -2(n-1)\Big(p_1(n-1)+p_0\Big)+6(n-2)\Big(p_1(n-2)+p_0\Big)+500(t+1)\cdot(n+1).$$

A bit of algebra on both sides and we find

$$4p_1n^2+4p_0n = 4p_1n^2+\Big(500(t+1)+4p_0-20p_1\Big)n+\Big(500(t+1)-10p_0+22p_1\Big).$$

Thus, we must solve

$$\begin{aligned} 4p_1 &= 4p_1 \\ 4p_0 &= 500(t+1)+4p_0-20p_1 \\ 0 &= 500(t+1)-10p_0+22p_1. \end{aligned}$$

we find

$$p_0 = 105(t+1), \qquad p_1 = 25(t+1).$$

Thus, we have

$$a_n^{(p)} = n\Big(25(t+1)n+105(t+1)\Big)\, 2^n.$$

(d) (2 marks) Find the general solution to the non-homogeneous linear recurrence.

Solution: In the following solution, replace t by the last digit of your student number and you will find your personnalized solution!
From a theorem seen in class, the general solution is

$$\begin{aligned} a_n &= a_n^{(h)}+a_n^{(p)} \\ &= \alpha_1\cdot 2^n+\alpha_2\cdot(-3)^n+n\Big(25(t+1)n+105(t+1)\Big)\, 2^n, \end{aligned}$$

for some constants α_1 and α_2.
We find the values of α_1 and α_2 by considering the initial values. We must solve

$$0 = a_0 = \alpha_1\cdot 2^0+\alpha_2\cdot(-3)^0+0\cdot\Big(25(t+1)\cdot 0+105(t+1)\Big)\, 2^0 = \alpha_1+\alpha_2$$
$$0 = a_1 = \alpha_1\cdot 2^1+\alpha_2\cdot(-3)^1+1\cdot\Big(25(t+1)\cdot 1+105(t+1)\Big)\, 2^1 = 2\alpha_1-3\alpha_2+260(t+1).$$

We find

$$\alpha_1 = -52(t+1), \qquad \alpha_2 = 52(t+1).$$

Thus, we have

$$a_n = -52(t+1)\cdot 2^n+52(t+1)\cdot(-3)^n+n\Big(25(t+1)n+105(t+1)\Big)\, 2^n.$$

Note: You must provide all calculations and justifications for parts (b), (c) and (d).

2. (10 marks) Use the Master Theorem to solve the following recurrence.

$$f(n) = 8 \cdot f(n/2) + 2020 \cdot n^3 \qquad (n > 1)$$
$$f(1) = 1$$

Note: You must provide all justifications.

SOLUTION: We have $a = 8$, $b = 2$, $c = 2020$ and $d = 3$. Thus, we have $d = 3 = \log_2(8) = \log_b(a)$. Thus, we are in the second case of the Master Theorem. Thus, $f(n) = O\left(n^d \log(n)\right) = O\left(n^3 \log(n)\right)$.