Chapter 1
Solutions

1. It would return $i = 2$ and $j = 3$

   Warning ! We cannot use 100 as the first term since we must have $i < j$.

3. It would return $i = 3$ and $j = 1$

5. In all cases, it would return $i = 1$ and $j = 3$.
7. The reasoning is as follows.

$$\begin{array}{l} (\neg P \vee \neg B) \to (V \wedge T) \\ \neg T \\ \hline P \end{array}$$

   where

$$\begin{aligned} P &= \text{“It rained.”} \\ B &= \text{“There is fog.”} \\ V &= \text{“The sailing race took place.”} \\ T &= \text{“The trophy was awarded.”} \end{aligned}$$

   We therefore want to know if $(\neg P \vee \neg B) \to (V \wedge T)$ and $\neg T$ imply P. The answer is yes. In the truth table of

$$\big((\neg P \vee \neg B) \to (V \wedge T)\big) \wedge \neg T \to P,$$

   we find “TRUE” in each line.
9. The statement is of the form

$$\forall x \; \exists y \; \exists z \; (x = y^2 - z^2),$$

   where the domain of x is the set of all odd integers, the domain of y is the set of all integers and the domain of z is the set of all integers.

   This statement is true. Here is a direct proof.
   Let x be an arbitrary odd integer. We therefore have $x = 2k + 1$ for an integer k. Therefore

$$x = 2k + 1 = k^2 + 2k + 1 - k^2 = (k + 1)^2 - (k)^2$$

corresponds to the difference of two squares.

We could just as well say that the statement is of the form

$$\forall x \ \left(x \textit{ is odd} \ \rightarrow \ \exists y \, \exists z \ (x = y^2 - z^2)\right),$$

where the domain of x, of y and of z is the set of all integers. It does not change the proof.

11. The statement is of the form

$$\forall n (I(n^3 + 5) \rightarrow \neg I(n)),$$

where the domain of n is the set of all integers and $I(n) =$ "n is odd."

This statement is true. Here is a proof by contrapositive.
Let n be an integer. We will show that $I(n) \rightarrow \neg I(n^3 + 5)$. Suppose that n is odd, therefore $n = 2k + 1$ for an integer k. We find

$$\begin{aligned}
& n^3 + 5 \\
=& (2k+1)^3 + 5 \\
=& (4k^2 + 4k + 1)(2k + 1) + 5 \\
=& (8k^3 + 12k^2 + 6k + 1) + 5 \\
=& 8k^3 + 12k^2 + 6k + 6 \\
=& 2(4k^3 + 6k^2 + 3k + 3)
\end{aligned}$$

which is even.

13. The statement is of the form

$$\exists A \ (A \text{ has thirteen "Friday the 13ths".}),$$

where the domain of A is the set of all years.

This statement is false. Here is a direct proof.
There can be a maximum of one "Friday the 13th" per month. There are twelve months. Therefore there cannot be more than twelve "Friday the 13ths" per year.

15. The statement is of the form

$$\forall x (I(x + 5) \rightarrow \neg I(x^2)),$$

where the domain of x is the set of all integers and $I(x) =$ "x is odd."

This statement is true. Here is a direct proof. Since $x + 5$ is odd, then $x + 5 = 2k + 1$ for an integer k. Therefore we have

$$\begin{aligned} x + 5 &= 2k + 1 \\ x &= 2k - 4 \\ x^2 &= (2k - 4)^2 \\ x^2 &= 4k^2 - 16k + 16 \\ x^2 &= 2(k^2 - 8k + 8) \end{aligned}$$

Which is even.

17. The form of this statement is a simple proposition

The number $\sqrt{5}$ is irrational.

This statment is true. Here is a proof by contradiction.

Suppose that $\sqrt{5}$ is rational. Therefore, we have

$$\sqrt{5} = \frac{a}{b}$$

for two integers a and b such that $b \neq 0$ and the fraction $\frac{a}{b}$ is reduced. Therefore

$$\begin{aligned} \sqrt{5} &= \frac{a}{b} \\ 5 &= \frac{a^2}{b^2} \\ 5b^2 &= a^2 \end{aligned}$$

Therefore a^2 is divisible by 5. This implies that a is divisible by 5. (In class we showed that a^2 is even, so a is even. Here, we use the fact that if a^2 is divisible by 5, then a is divisible by 5. You must prove it yourself to be convinced...) Since a is divisible by 5, we have $a = 5d$ for an integer d. Therefore

$$\begin{aligned} 5b^2 &= a^2 \\ 5b^2 &= (5d)^2 \\ 5b^2 &= 5^2 d^2 \\ b^2 &= 5d^2 \end{aligned}$$

Therefore b^2 is divisible by 5. This implies that b is divisible by 5.
We found 5 as a common divisor between a and b, which implies that the fraction $\frac{a}{b}$ is not reduced. This is a contradiction.

19. The form of this statement is a simple proposition.

The number $\log_2(3)$ is irrational.

This statement is true. Here is a proof by contradiction.

Suppose that $\log_2(3)$ is rational. Since $\log_2(3) > 1$, there exist two integers $p \geq 0$ and $q > 0$ such that $\log_2(3) = \frac{p}{q}$. We obtain

$$\begin{aligned} \log_2(3) &= \frac{p}{q} \\ 3 &= 2^{p/q} \\ 3^q &= 2^p. \end{aligned}$$

If $p \neq 0$, then 2^p is even. It's impossible since 3^q is odd and $3^q = 2^p$. So $p = 0$. But $\log_2(3) = 0$, which is a contradiction.

21. This statement is of the form

$$\forall n \left(\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \right),$$

where the domain of n is $\mathbf{N}$.
This statement is true. Here is a proof by induction.

Base case : For $n = 0$,

$$0 = \frac{0 \cdot (0+1) \cdot (2 \cdot 0 + 1)}{6}.$$

Induction hypothesis : Let $k \geq 0$. Suppose that

$$\sum_{i=0}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}.$$

Induction step : For $n = k+1$, we have

$$\begin{aligned}
&\sum_{i=0}^{k+1} i^2 \\
&= \sum_{i=0}^{k} i^2 + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 && \text{induction hypothesis} \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{(k+1)(2k^2+k+6k+6)}{6} \\
&= \frac{(k+1)(2k^2+7k+6)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} \\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}
\end{aligned}$$

23. The statement is of the form

$$\forall n \left(\sum_{i=0}^{n} 3 \cdot 5^i = \frac{3(5^{n+1}-1)}{4} \right),$$

where the domain of n is $\mathbf{N}$.
This statement is true. Here is a proof by induction.

Base case : For $n = 0$,

$$3 \cdot 5^0 = 3 = \frac{3(5^{0+1}-1)}{4}.$$

Induction hypothesis : Let $k \geq 0$. Suppose that

$$\sum_{i=0}^{k} 3 \cdot 5^i = \frac{3(5^{k+1}-1)}{4}.$$

Induction step : For $n = k+1$, we have

$$\begin{aligned}
&\sum_{i=0}^{k+1} 3\cdot 5^i \\
=&\sum_{i=0}^{k} 3\cdot 5^i + 3\cdot 5^{k+1} \\
=&\frac{3(5^{k+1}-1)}{4} + 3\cdot 5^{k+1} && \text{Induction hypthesis} \\
=&\frac{3(5^{k+1}-1) + 4\cdot 3\cdot 5^{k+1}}{4} \\
=&\frac{3\cdot 5^{k+1} + 12\cdot 5^{k+1} - 3}{4} \\
=&\frac{15\cdot 5^{k+1} - 3}{4} \\
=&\frac{3\cdot 5^{k+2} - 3}{4} \\
=&\frac{3\cdot\left(5^{(k+1)+1} - 1\right)}{4}
\end{aligned}$$

Here is a completely different proof. It is a direct proof. Let's first write the sum under the form of a sum s. We have

$$s = \sum_{i=0}^{n} 3\cdot 5^i = 3(1+5+5^2+\cdots+5^n).$$

We therefore find

$$\frac{s}{3} = 1+5+5^2+\cdots+5^n,$$

where

$$\frac{s}{3} - 1 = 5+5^2+\cdots+5^n = 5\cdot\left(1+5+\cdots+5^{n-1}\right) = 5\cdot\left(\frac{s}{3} - 5^n\right)$$

We have an equation that lets us isolate an expression for s :

$$\begin{aligned}
\frac{s}{3} - 1 &= 5\cdot\left(\frac{s}{3} - 5^n\right) \\
\frac{s}{3} - 1 &= 5\cdot\frac{s}{3} - 5^{n+1} \\
5^{n+1} - 1 &= 4\cdot\frac{s}{3} \\
s &= \frac{3}{4}(5^{n+1} - 1).
\end{aligned}$$

25. This statement is of the form

$$\forall n(n^3 - n \text{ is divisible by } 3),$$

where the domain of n is $\mathbb{N}$.
This statement is true. Here is a proof by induction.
Base case : For $n = 0$,

$$0^3 - 0 = 0 = 3 \cdot 0$$

is divisible by 3.
Induction hypothesis : Let $k \geq 0$. Suppose that $k^3 - k$ is divisible by 3.
Induction step : By the induction hypothesis, $k^3 - k$ is divisible by 3. Therefore $k^3 - k = 3\ell$ for an integer ℓ.
For $n = k + 1$, we have that

$$(k+1)^3-(k+1) = k^3+3k^2+3k+1-k-1 = (k^3-k)+(3k^2+3k) = 3\ell+3(k^2+k) = 3(\ell+k^2+k)$$

is divisible by 3.
Here is another completely different proof. This is a direct proof. We have

$$n^3 - n = n(n^2 - 1) = n(n + 1)(n - 1).$$

We consider three cases : (i) $n = 3k$, (ii) $n = 3k + 1$, (iii) $n = 3k + 2$. (i) If $n = 3k$ then

$$\begin{aligned} n^3 - n &= n(n + 1)(n - 1) \\ &= (3k)(3k + 1)(3k - 1) \\ &= 3 \cdot [k(3k + 1)(3k - 1)] \end{aligned}$$

(ii) If $n = 3k + 1$ then

$$\begin{aligned} n^3 - n &= n(n + 1)(n - 1) \\ &= (3k + 1)(3k + 2)(3k) \\ &= 3 \cdot [(3k + 1)(3k + 2)k] \end{aligned}$$

(iii) If $n = 3k + 2$ then

$$\begin{aligned} n^3 - n &= n(n + 1)(n - 1) \\ &= (3k + 2)(3k + 3)(3k + 1) \\ &= 3 \cdot [(3k + 2)(k + 1)(3k + 1)] \end{aligned}$$

27. (a) P is true for all odd integers.
    (b) P is true for all positive integers.
    (c) P is true for all powers of 2.
    (d) P is true for all positive integers.