Chapter 4.3
Solutions

1.

(a) $2^3 \cdot 11$
(b) 3^6
(c) $7 \cdot 11 \cdot 13$
(d) $2 \cdot 3^3 \cdot 5 \cdot 7 \cdot 13 \cdot 37$

3.

Here is small Python snippet to obtain the solution.

```
primes = []
for n in range(2, 2020):
    for p in primes:
        if n % p == 0:
            break
    else:
        primes.append(n)
print(len(primes))
```

Answer : 306

5.

We have two integers a, b such that the product $ab = 2^7 \cdot 3^8 \cdot 5^2 \cdot 7^{11}$ and $\gcd(a, b) = 2^3 \cdot 3^4 \cdot 5$. Then their least common multiple is

$$\begin{aligned} \frac{ab}{\gcd(a,b)} &= \frac{2^7 \cdot 3^8 \cdot 5^2 \cdot 7^{11}}{2^3 \cdot 3^4 \cdot 5} \\ &= 2^4 \cdot 3^4 \cdot 5 \cdot 7^{11} \end{aligned}$$

7.

(a) $3s + 4t = 1$

$$\begin{aligned} 4 &= 3 \cdot 1 + 1 \\ 3 &= 1 \cdot 3 + 0 \\ 1 &= 4 - 3 \\ 1 &= 4 \cdot (1) + 3 \cdot (-1) \end{aligned}$$

$s = -1, t = 1$

(b) $1534s + 2020t = 1608$
$767s + 1010t = 804$

$$
\begin{aligned}
1010 &= 767 \cdot 1 + 243 \\
767 &= 243 \cdot 3 + 38 \\
243 &= 38 \cdot 6 + 15 \\
38 &= 15 \cdot 2 + 8 \\
15 &= 8 \cdot 1 + 7 \\
8 &= 7 \cdot 1 + 1 \\
7 &= 1 \cdot 7 + 0 \\
1 &= 8 - 7 \\
1 &= (38 - 15 \cdot 2) - (15 - 8) \\
1 &= 38 - 15 \cdot 3 + 8 \\
1 &= (767 - 243 \cdot 3) - (243 - 38 \cdot 6) \cdot 3 + (38 - 15 \cdot 2) \\
1 &= 767 - 243 \cdot 3 - 243 \cdot 3 + 38 \cdot 18 + 38 - 15 \cdot 2 \\
1 &= 767 - 243 \cdot 6 + 38 \cdot 19 - 15 \cdot 2 \\
1 &= 767 - (1010 - 767) \cdot 6 + (767 - 243 \cdot 3) \cdot 19 - (243 - 38 \cdot 6) \cdot 2 \\
1 &= 767 - 1010 \cdot 6 + 767 \cdot 6 + 767 \cdot 19 - 243 \cdot 57 - 243 \cdot 2 + 38 \cdot 12 \\
1 &= -1010 \cdot 6 + 767 \cdot 26 - 243 \cdot 59 + 38 \cdot 12 \\
1 &= -1010 \cdot 6 + 767 \cdot 26 - (1010 - 767) \cdot 59 + (767 - 243 \cdot 3) \cdot 12 \\
1 &= -1010 \cdot 6 + 767 \cdot 26 - 1010 \cdot 59 + 767 \cdot 59 + 767 \cdot 12 - 243 \cdot 36 \\
1 &= -1010 \cdot 65 + 767 \cdot 97 - 243 \cdot 36 \\
1 &= -1010 \cdot 65 + 767 \cdot 97 - (1010 - 767) \cdot 36 \\
1 &= -1010 \cdot 65 + 767 \cdot 97 - 1010 \cdot 36 + 767 \cdot 36 \\
1 &= -1010 \cdot 101 + 767 \cdot 133 \\
1 &= 1010 \cdot (-101) + 767 \cdot (133) \\
804 &= 1010 \cdot (-101 \cdot 804) + 767 \cdot (133 \cdot 804) \\
804 &= 1010 \cdot (-81204) + 767 \cdot (106932) \\
1608 &= 2020 \cdot (-81204) + 1534 \cdot (106932)
\end{aligned}
$$

$s = 106932, t = -81204$

(c) $25s + 36t = 3$

$$
\begin{aligned}
36 &= 25 \cdot 1 + 11 \\
25 &= 11 \cdot 2 + 3 \\
11 &= 3 \cdot 3 + 2 \\
3 &= 2 \cdot 1 + 1 \\
2 &= 1 \cdot 2 + 0 \\
1 &= 3 - 2 \\
1 &= (25 - 11 \cdot 2) - (11 - 3 \cdot 3) \\
1 &= 25 - 11 \cdot 2 - 11 + 3 \cdot 3 \\
1 &= 25 - 11 \cdot 3 + 3 \cdot 3 \\
1 &= 25 - (36 - 25) \cdot 3 + (25 - 11 \cdot 2) \cdot 3 \\
1 &= 25 \cdot 7 - 36 \cdot 3 - 11 \cdot 6 \\
1 &= 25 \cdot 7 - 36 \cdot 3 - (36 - 25) \cdot 6 \\
1 &= 25 \cdot 7 - 36 \cdot 3 - 36 \cdot 6 + 25 \cdot 6 \\
1 &= 25 \cdot 13 - 36 \cdot 9 \\
1 &= 25 \cdot 13 + 36 \cdot -9 \\
3 &= 25 \cdot (39) + 36 \cdot (-27)
\end{aligned}
$$

$s = 39, t = -27$

(d) $17s + 31t = 1$

$$
\begin{aligned}
31 &= 17 \cdot 1 + 14\\
17 &= 14 \cdot 1 + 3\\
14 &= 3 \cdot 4 + 2\\
3 &= 2 \cdot 1 + 1\\
2 &= 1 \cdot 2 + 0\\
1 &= 3 - 2\\
1 &= (17 - 14) - (14 - 3 \cdot 4)\\
1 &= 17 - 14 - 14 + 3 \cdot 4\\
1 &= 17 - 14 \cdot 2 + 3 \cdot 4\\
1 &= 17 - (31 - 17) \cdot 2 + (17 - 14) \cdot 4\\
1 &= 17 - 31 \cdot 2 + 17 \cdot 2 + 17 \cdot 4 - 14 \cdot 4\\
1 &= 17 \cdot 7 - 31 \cdot 2 - 14 \cdot 4\\
1 &= 17 \cdot 7 - 31 \cdot 2 - (31 - 17) \cdot 4\\
1 &= 17 \cdot 7 - 31 \cdot 2 - 31 \cdot 4 + 17 \cdot 4\\
1 &= 17 \cdot 11 + (31) \cdot (-6)
\end{aligned}
$$

$s = 11, t = -6$

9.

The statement is of the form

$$\forall n\Big(\exists p\big(p|n \wedge P(p) \wedge p \leq \sqrt{n}\big) \rightarrow \neg P(n)\Big),$$

where the domain of n is the set of all integers greater than 1, the domain of p is the set of positive integers and $P(n) =$ "n is prime".

The statement is true, here is a direct proof.

Let $n \geq 1$ be an integer. Suppose there exists a prime number p such that $p|n$ and $p \leq \sqrt{n}$.

Since p is prime, $p \geq 2$. Then p divides n and $2 \leq p \leq \sqrt{n}$. Then n is divisible by a number that is not 1 or itself. n therefore admits more than two divisors. Therefore n is not prime.

11.

The statment is of the form

$$\forall a \forall b \forall q \forall r\ (a = bq + r \rightarrow \gcd(a, b) = \gcd(b, r))$$

where the domain of a, b, q and r is $\mathbb{Z}$.

The statement is true, here is a direct proof : lemma seen in class !

13.

The statement is of the form

$$\forall a \forall b \forall c\ ((\gcd(a, b) = 1) \wedge a|bc \rightarrow a|c)$$

where the domain of a, b and c is the set of positive integers.

The statement is true, here is a direct proof : lemma seen in class !

15.

The statement is of the form

$$\forall a \forall b \forall c\ ((\gcd(a, b) = 1) \wedge a|c \rightarrow a|bc)$$

where the domain of a, b and c is the set of positive integers.

The statement is true, here is a direct proof.

Let a, b and c be positive integers. Suppose that $\gcd(a, b) = 1$ and that $a|c$. By theorem 1 (part 2), we have $a|bc$.

Caution ! In this proof, we do not need to use the hypothesis "$\gcd(a, b) = 1$". The statement would be true even without the hypothesis !