Chapter 8

1. (a) The characteristic equation is $r^2 - r - 6 = (r-3)(r+2)$. Therefore, the general solution is
$$a_n = \alpha_1 \cdot 3^n + \alpha_2 \cdot (-2)^n.$$
We have
$$\begin{aligned} 0 &= a_0 = \alpha_1 + \alpha_2 \\ 1 &= a_1 = 3\alpha_1 - 2\alpha_2. \end{aligned}$$
So $\alpha_1 = \frac{1}{5}$ and $\alpha_2 = -\frac{1}{5}$. Thus
$$a_n = \frac{1}{5} \cdot 3^n - \frac{1}{5} \cdot (-2)^n.$$

(b) The characteristic equation is $r^2 + 3r + 2 = (r+2)(r+1)$. Therefore, the general solution is
$$a_n = \alpha_1 \cdot (-2)^n + \alpha_2 \cdot (-1)^n.$$
We have
$$\begin{aligned} 0 &= a_0 = \alpha_1 + \alpha_2 \\ 1 &= a_1 = -2\alpha_1 - \alpha_2. \end{aligned}$$
So $\alpha_1 = -1$ and $\alpha_2 = 1$. Thus
$$a_n = -(-2)^n + (-1)^n.$$

(c) The characteristic equation is $r^2 + 5r - 6 = (r+6)(r-1)$. Therefore, the general solution is
$$a_n = \alpha_1 \cdot (-6)^n + \alpha_2 \cdot 1^n = \alpha_1 \cdot (-6)^n + \alpha_2.$$
We have
$$\begin{aligned} 0 &= a_0 = \alpha_1 + \alpha_2 \\ 1 &= a_1 = -6\alpha_1 + \alpha_2. \end{aligned}$$
So $\alpha_1 = -\frac{1}{7}$ and $\alpha_2 = \frac{1}{7}$. Thus
$$a_n = -\frac{1}{7} \cdot (-6)^n + \frac{1}{7}.$$

3. (a) The characteristic equation is $r^2-10r+21=(r-3)(r-7)$. Therefore, the general solution is

$$a_n = \alpha_1 \cdot 3^n + \alpha_2 \cdot 7^n.$$

We have

$$\begin{aligned} 0 &= a_0 = \alpha_1 + \alpha_2 \\ 1 &= a_1 = 3\alpha_1 + 7\alpha_2. \end{aligned}$$

So $\alpha_1 = -\frac{1}{4}$ and $\alpha_2 = \frac{1}{4}$. Thus

$$a_n = -\frac{1}{4} \cdot 3^n + \frac{1}{4} \cdot 7^n.$$

(b) The characteristic equation is $r^2 - 1 = (r+1)(r-1)$. Therefore, the general solution is

$$a_n = \alpha_1 \cdot (-1)^n + \alpha_2 \cdot 1^n = \alpha_1 \cdot (-1)^n + \alpha_2.$$

We have

$$\begin{aligned} 0 &= a_0 = \alpha_1 + \alpha_2 \\ 1 &= a_1 = -\alpha_1 + \alpha_2. \end{aligned}$$

So $\alpha_1 = -\frac{1}{2}$ and $\alpha_2 = \frac{1}{2}$. Thus

$$a_n = -\frac{1}{2} \cdot (-1)^n + \frac{1}{2}.$$

(c) The characteristic equation is $r^2-6r+9=(r-3)(r-3)$. Therefore, the general solution is

$$a_n = \alpha_1 \cdot 3^n + \alpha_2 n \cdot 3^n.$$

We have

$$\begin{aligned} 0 &= a_0 = \alpha_1 \\ 1 &= a_1 = 3\alpha_1 + 3\alpha_2. \end{aligned}$$

So $\alpha_1 = 0$ and $\alpha_2 = \frac{1}{3}$. Thus

$$a_n = \frac{1}{3} n \cdot 3^n.$$

5. Since $r^2 - 2r = r(r-2)$, we find that the two characteristic roots are 0 and 2. So the homogeneous solution is of the form

$$\alpha_1 \cdot 0^n + \alpha_2 \cdot 2^n = \alpha_2 \cdot 2^n.$$

7. After testing a few values, we see that the sequence of a_i gives us

$$0, 1, 0, -1, 0, 1, 0, -1, 0, 1, 0, -1, 0, 1, 0, -1, \ldots.$$

We can easily prove it by induction (proof left to the reader!). We have a rule, it's solved.

Alternately, we can use the method shown in class. We don't need induction but we find we need complex numbers!

The characteristic polynomial is $r^2 + 1 = (r + \sqrt{-1})(r - \sqrt{-1}) = (r + I)(r - I)$. So the solution is

$$a_n = \alpha_1 \cdot I^n + \alpha_2 \cdot (-I)^n.$$

We have

$$\begin{aligned} 0 = a_0 &= \alpha_1 \cdot I^0 + \alpha_2 \cdot (-I)^0 = \alpha_1 + \alpha_2 \\ 1 = a_1 &= \alpha_1 \cdot I^1 + \alpha_2 \cdot (-I)^1 = \alpha_1 \cdot I - \alpha_2 \cdot I \end{aligned}$$

So $\alpha_1 = -\frac{I}{2}$ and $\alpha_2 = \frac{I}{2}$.

$$a_n = -\frac{I}{2} \cdot I^n + \frac{I}{2} \cdot (-I)^n.$$

9. Solve the following recurrence relations.

   (a) The characteristic equation of the associated homogeneous equation is $r^2+3r+2 = (r+1)(r+2)$. So the solution of the associated homogeneous equation is

   $$a_n^{(h)} = \alpha_1 \cdot (-1)^n + \alpha_2 \cdot (-2)^n.$$

   A particular solution is of the form $(p_1 n + p_0)2^n$ for constants p_0 and p_1. We have

   $$\begin{aligned} (p_1 n + p_0)2^n &= -3(p_1(n-1) + p_0)2^{n-1} - 2(p_1(n-2) + p_0)2^{n-2} + n2^n \\ 4(p_1 n + p_0) &= -6(p_1(n-1) + p_0) - 2(p_1(n-2) + p_0) + 4n, \end{aligned}$$

   and so

   $$4p_1 n + 4p_0 = (4 - 8p_1)n + (-8p_0 + 10p_1),$$

   which gives us

   $$\begin{aligned} 4p_1 &= 4 - 8p_1 \\ 4p_0 &= -8p_0 + 10p_1. \end{aligned}$$

   We find $p_0 = \frac{5}{18}$ et $p_1 = \frac{1}{3}$. So

   $$a_n^{(p)} = \left(\frac{1}{3}n + \frac{5}{18}\right)2^n.$$

Therefore

$$a_n = a_n^{(h)} + a_n^{(p)} = \alpha_1 \cdot (-1)^n + \alpha_2 \cdot (-2)^n + \left(\frac{1}{3}n + \frac{5}{18}\right) 2^n.$$

We have

$$\begin{aligned}
0 = a_0 = \alpha_1 \cdot (-1)^0 + \alpha_2 \cdot (-2)^0 + \left(\frac{1}{3} \cdot 0 + \frac{5}{18}\right) 2^0 = \alpha_1 + \alpha_2 + \frac{5}{18} \\
1 = a_1 = \alpha_1 \cdot (-1)^1 + \alpha_2 \cdot (-2)^1 + \left(\frac{1}{3} \cdot 1 + \frac{5}{18}\right) 2^1 = -\alpha_1 - 2\alpha_2 + \frac{11}{9}
\end{aligned}$$

More compactly, we have the system of equations

$$\begin{aligned}
\alpha_1 + \alpha_2 &= -\frac{5}{18} \\
\alpha_1 + 2\alpha_2 &= \frac{2}{9}
\end{aligned}$$

So $\alpha_1 = -\frac{7}{9}$ and $\alpha_2 = \frac{1}{2}$, with which we find the final solution :

$$a_n = -\frac{7}{9} \cdot (-1)^n + \frac{1}{2} \cdot (-2)^n + \left(\frac{1}{3}n + \frac{5}{18}\right) 2^n.$$

(b) The characteristic equation of the associated homogeneous equation is $r^3 - 6r^2 + 12r - 8 = (r-2)^3$. So the solution of the associated homogeneous equation is

$$a_n^{(h)} = \alpha_1 \cdot 2^n + \alpha_2 \cdot n \cdot 2^n + \alpha_3 \cdot n^2 \cdot 2^n.$$

A particular solution is of the form $n^3(p_1 n + p_0)2^n$ for constants p_0 and p_1. We have

$$\begin{aligned}
&n^3(p_1 n + p_0)2^n \\
&= 6(n-1)^3(p_1(n-1) + p_0)2^{n-1} \\
&\quad - 12(n-2)^3(p_1(n-2) + p_0)2^{n-2} \\
&\quad + 8(n-3)^3(p_1(n-3) + p_0)2^{n-3} \\
&\quad + (n+2)2^n,
\end{aligned}$$

from which we find

$$\begin{aligned}
&8n^3(p_1 n + p_0) \\
&= 24(n-1)^3(p_1(n-1) + p_0) \\
&\quad - 24(n-2)^3(p_1(n-2) + p_0) \\
&\quad + 8(n-3)^3(p_1(n-3) + p_0) \\
&\quad + 8(n+2),
\end{aligned}$$

and so

$$8p_1n^4 + 8p_0n^3 = 8p_1n^4 + 8p_0n^3 + (8 - 192p_1)n + (16 - 48p_0 + 288p_1),$$

which gives us

$$\begin{aligned} 8p_1 &= 8p_1 \\ 8p_0 &= 8p_0 \\ 0 &= 8 - 192p_1 \\ 0 &= 16 - 48p_0 + 288p_1. \end{aligned}$$

We find $p_0 = \frac{7}{12}$ and $p_1 = \frac{1}{24}$. So

$$a_n^{(p)} = n^3 \left(\frac{1}{24} n + \frac{7}{12} \right) 2^n.$$

Therefore

$$a_n = a_n^{(h)} + a_n^{(p)} = \alpha_1 \cdot 2^n + \alpha_2 \cdot n \cdot 2^n + \alpha_3 \cdot n^2 \cdot 2^n + n^3 \left(\frac{1}{24} n + \frac{7}{12} \right) 2^n$$

We have

$$\begin{aligned} 0 = a_0 &= \alpha_1 \cdot 2^0 + \alpha_2 \cdot 0 \cdot 2^0 + \alpha_3 \cdot 0^2 \cdot 2^0 + 0^3 \left(\frac{1}{24} \cdot 0 + \frac{7}{12} \right) 2^0 = \alpha_1 \\ 1 = a_1 &= \alpha_1 \cdot 2^1 + \alpha_2 \cdot 1 \cdot 2^1 + \alpha_3 \cdot 1^2 \cdot 2^1 + 1^3 \left(\frac{1}{24} \cdot 1 + \frac{7}{12} \right) 2^1 = 2\alpha_1 + 2\alpha_2 + 2\alpha_3 + \frac{5}{4} \\ 2 = a_2 &= \alpha_1 \cdot 2^2 + \alpha_2 \cdot 2 \cdot 2^2 + \alpha_3 \cdot 2^2 \cdot 2^2 + 2^3 \left(\frac{1}{24} \cdot 2 + \frac{7}{12} \right) 2^2 = 4\alpha_1 + 8\alpha_2 + 16\alpha_3 + \frac{64}{3}. \end{aligned}$$

More compactly, we have the system of equations

$$\begin{aligned} \alpha_1 &= 0 \\ 2\alpha_1 + 2\alpha_2 + 2\alpha_3 &= -\frac{1}{4} \\ 4\alpha_1 + 8\alpha_2 + 16\alpha_3 &= -\frac{58}{3} \end{aligned}$$

So $\alpha_1 = 0$, $\alpha_2 = \frac{13}{6}$ and $\alpha_3 = -\frac{55}{24}$, with which we find the final solution :

$$a_n = \frac{13}{6} \cdot n \cdot 2^n - \frac{55}{24} \cdot n^2 \cdot 2^n + n^3 \left(\frac{1}{24} n + \frac{7}{12} \right) 2^n.$$

(c) The characteristic equation of the associated homogeneous equation is $r^2+2r+1 = (r+1)^2$. So the solution of the associated homogeneous equation is

$$a_n^{(h)} = \alpha_1 \cdot (-1)^n + \alpha_2 \cdot n \cdot (-1)^n.$$

A particular solution is of the form $(p_2n^2 + p_1n + p_0)2^n$ for constants p_0, p_1 and p_2. We have

$$
\begin{aligned}
&(p_2n^2 + p_1n + p_0)2^n \\
=& -2(p_2(n-1)^2 + p_1(n-1) + p_0)2^{n-1} - (p_2(n-2)^2 + p_1(n-2) + p_0)2^{n-2} + (n^2+1)2^n,
\end{aligned}
$$

from which we find

$$
\begin{aligned}
&4(p_2n^2 + p_1n + p_0) \\
=& -4(p_2(n-1)^2 + p_1(n-1) + p_0) - (p_2(n-2)^2 + p_1(n-2) + p_0) + 4(n^2+1)
\end{aligned}
$$

and so

$$4p_2n^2 + 4p_1n + 4p_0 = (4 - 5p_2)n^2 + (-5p_1 + 12p_2)n + (4 - 5p_0 + 6p_1 - 8p_2),$$

which gives us

$$
\begin{aligned}
4p_2 &= 4 - 5p_2 \\
4p_1 &= -5p_1 + 12p_2 \\
4p_0 &= 4 - 5p_0 + 6p_1 - 8p_2.
\end{aligned}
$$

We find $p_0 = \frac{4}{9}$, $p_1 = \frac{16}{27}$ and $p_2 = \frac{4}{9}$. So

$$a_n^{(p)} = \left(\frac{4}{9}n^2 + \frac{16}{27}n + \frac{4}{9}\right)2^n.$$

And then

$$a_n = a_n^{(h)} + a_n^{(p)} = \alpha_1 \cdot (-1)^n + \alpha_2 \cdot n \cdot (-1)^n + \left(\frac{4}{9}n^2 + \frac{16}{27}n + \frac{4}{9}\right)2^n.$$

We have

$$
\begin{aligned}
0 = a_0 = \alpha_1 \cdot (-1)^0 + \alpha_2 \cdot 0 \cdot (-1)^0 + \left(\frac{4}{9} \cdot 0^2 + \frac{16}{27} \cdot 0 + \frac{4}{9}\right)2^0 &= \alpha_1 + \frac{4}{9} \\
1 = a_1 = \alpha_1 \cdot (-1)^1 + \alpha_2 \cdot 1 \cdot (-1)^1 + \left(\frac{4}{9} \cdot 1^2 + \frac{16}{27} \cdot 1 + \frac{4}{9}\right)2^1 &= -\alpha_1 - \alpha_2 + \frac{80}{27}.
\end{aligned}
$$

More compactly, we have the system of equations

$$
\begin{aligned}
\alpha_1 + \frac{4}{9} &= 0 \\
\alpha_1 + \alpha_2 &= \frac{53}{27}.
\end{aligned}
$$

So $\alpha_1 = -\frac{4}{9}$ and $\alpha_2 = \frac{65}{27}$, with which we find the final solution :

$$a_n = -\frac{4}{9} \cdot (-1)^n + \frac{65}{27} \cdot n \cdot (-1)^n + \left(\frac{4}{9}n^2 + \frac{16}{27}n + \frac{4}{9}\right)2^n.$$

(d) The characteristic equation of the associated homogeneous equation is $r^3 - 3r^2 + 4 = (r-2)^2(r+1)$. So the solution of the associated homogeneous equation is

$$a_n^{(h)} = \alpha_1 \cdot (-1)^n + \alpha_2 \cdot 2^n + \alpha_3 \cdot n \cdot 2^n.$$

A particular solution is of the form $n^2(p_1 n + p_0)2^n$ for constants p_0 nd p_1. We have

$$\begin{aligned} n^2(p_1 n + p_0)2^n &= 3(n-1)^2(p_1(n-1)+p_0)2^{n-1} - 4(n-3)^2(p_1(n-3)+p_0)2^{n-3} - (n+4)2^n \\ 8n^2(p_1 n + p_0) &= 12(n-1)^2(p_1(n-1)+p_0) - 4(n-3)^2(p_1(n-3)+p_0) - 8(n+4) \\ 8p_1 n^3 + 8p_0 n^2 &= 8p_1 n^3 + 8p_0 n^2 + (-72p_1 - 8)n + (-32 - 24p_0 + 96p_1) \end{aligned}$$

So

$$\begin{aligned} 8p_1 &= 8p_1 \\ 8p_0 &= 8p_0 \\ 0 &= -72p_1 - 8 \\ 0 &= -32 - 24p_0 + 96p_1 \end{aligned}$$

from which we find $p_0 = -\frac{16}{9}$ and $p_1 = -\frac{1}{9}$ and

$$a_n^{(p)} = -n^2\left(\frac{1}{9}n + \frac{16}{9}\right)2^n.$$

Therefore

$$a_n = a_n^{(h)} + a_n^{(p)} = \alpha_1 \cdot (-1)^n + \alpha_2 \cdot 2^n + \alpha_3 \cdot n \cdot 2^n - n^2\left(\frac{1}{9}n + \frac{16}{9}\right)2^n.$$

We have

$$\begin{aligned} 0 = a_0 &= \alpha_1 \cdot (-1)^0 + \alpha_2 \cdot 2^0 + \alpha_3 \cdot 0 \cdot 2^0 - 0^2\left(\frac{1}{9}\cdot 0 + \frac{16}{9}\right)2^0 = \alpha_1 + \alpha_2 \\ 1 = a_1 &= \alpha_1 \cdot (-1)^1 + \alpha_2 \cdot 2^1 + \alpha_3 \cdot 1 \cdot 2^1 - 1^2\left(\frac{1}{9}\cdot 1 + \frac{16}{9}\right)2^1 = -\alpha_1 + 2\alpha_2 + 2\alpha_3 - \frac{34}{9} \\ 2 = a_2 &= \alpha_1 \cdot (-1)^2 + \alpha_2 \cdot 2^2 + \alpha_3 \cdot 2 \cdot 2^2 - 2^2\left(\frac{1}{9}\cdot 2 + \frac{16}{9}\right)2^2 = \alpha_1 + 4\alpha_2 + 8\alpha_3 - 32. \end{aligned}$$

More compactly, we have the system of equations

$$\begin{aligned} \alpha_1 + \alpha_2 &= 0 \\ -\alpha_1 + 2\alpha_2 + 2\alpha_3 &= \frac{43}{9} \\ \alpha_1 + 4\alpha_2 + 8\alpha_3 &= 34. \end{aligned}$$

So $\alpha_1 = \frac{134}{81}$, $\alpha_2 = -\frac{134}{81}$ and $\alpha_3 = \frac{263}{54}$, with which we find the final solution :

$$a_n = \frac{134}{81} \cdot (-1)^n - \frac{134}{81} \cdot 2^n + \frac{263}{54} \cdot n \cdot 2^n - n^2 \left(\frac{1}{9} n + \frac{16}{9} \right) 2^n.$$

11. Solve the following recurrence relations.

(a) The equation can be written

$$a_n = a_{n-1} + \left(\frac{1}{2} \right)^n,$$

where $a_1 = 1$. So we solve it the same way as the others.
The characteristic equation of the associated homogeneous equation is $r^2 - r - 0 = r(r-1)$. So the solution of the associated homogeneous equation is

$$a_n^{(h)} = \alpha_1 \cdot 0^n + \alpha_2 \cdot 1^n = \alpha_2.$$

A particular solution is of the form $c \cdot \left(\frac{1}{2}\right)^n$ for a constant c. We have

$$\begin{aligned} c \cdot \left(\frac{1}{2} \right)^n &= c \cdot \left(\frac{1}{2} \right)^{n-1} + \left(\frac{1}{2} \right)^n \\ c &= 2c + 1 \\ c &= -1. \end{aligned}$$

Therefore

$$a_n^{(p)} = -\left(\frac{1}{2} \right)^n.$$

So

$$a_n = a_n^{(h)} + a_n^{(p)} = \alpha_2 - \left(\frac{1}{2} \right)^n.$$

We have

$$1 = a_1 = \alpha_2 - \left(\frac{1}{2} \right)^1.$$

This gives $\alpha_2 = \frac{3}{2}$, hence the final solution :

$$a_n = \frac{3}{2} - \left(\frac{1}{2} \right)^n.$$

(b) The equation can be written

$$a_n = a_{n-1} + n \left(\frac{1}{2} \right)^n,$$

where $a_1 = 1$. So we solve it the same way as the others.
The characteristic equation of the associated homogeneous equation is $r^2 - r - 0 = r(r-1)$. So the solution of the associated homogeneous equation is

$$a_n^{(h)} = \alpha_1 \cdot 0^n + \alpha_2 \cdot 1^n = \alpha_2.$$

A particular solution is of the form $(p_1 \cdot n + p_0)\left(\frac{1}{2}\right)^n$ for constants p_0 et p_1. We have

$$\begin{aligned}(p_1 \cdot n + p_0)\left(\frac{1}{2}\right)^n &= (p_1 \cdot (n-1) + p_0)\left(\frac{1}{2}\right)^{n-1} + n\left(\frac{1}{2}\right)^n \\ p_1 \cdot n + p_0 &= 2p_1 \cdot (n-1) + 2p_0 + n \\ p_1 \cdot n + p_0 &= (2p_1 + 1)n + (2p_0 - 2p_1)\end{aligned}$$

Therefore

$$\begin{aligned}p_1 &= 2p_1 + 1 \\ p_0 &= 2p_0 - 2p_1,\end{aligned}$$

which gives us $p_0 = -2$ and $p_1 = -1$ and

$$a_n^{(p)} = -(n+2)\left(\frac{1}{2}\right)^n.$$

So

$$a_n = a_n^{(h)} + a_n^{(p)} = \alpha_2 - (n+2)\left(\frac{1}{2}\right)^n.$$

We have

$$1 = a_1 = \alpha_2 - (1+2)\left(\frac{1}{2}\right)^1.$$

Which gives $\alpha_2 = \frac{5}{2}$, and the final solution :

$$a_n = \frac{5}{2} - (n+2)\left(\frac{1}{2}\right)^n.$$