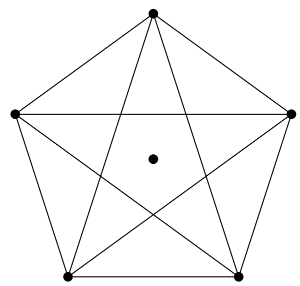
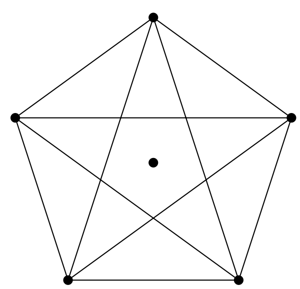
Graphs - Solutions

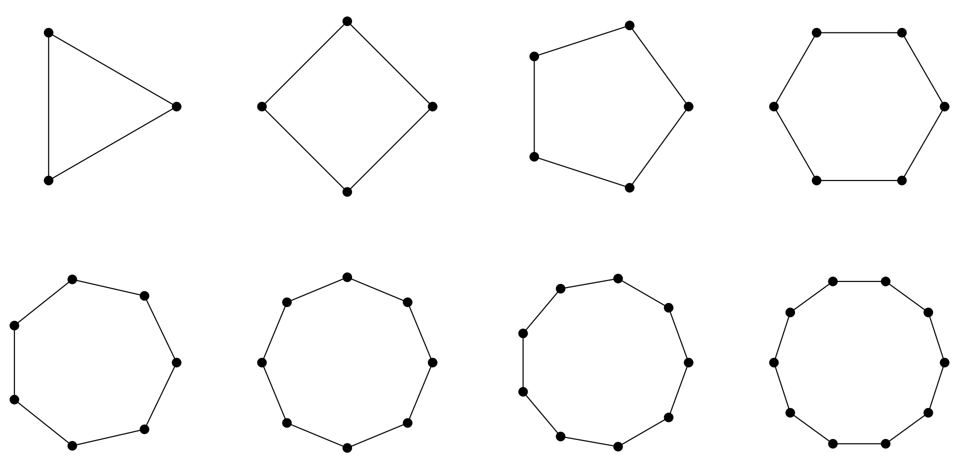
1. Let $G = (V, E)$ be a non-directed graph. Set $v = |V|$ and $e = |E|$. True or false ? Prove or disprove the following statements.
   (a) This is false. See the following counterexample.



   (b) This is true. We proved this result in class.
   (c) This is true. We proved this result in class.
   (d) This is false. See the following counterexample.



3.



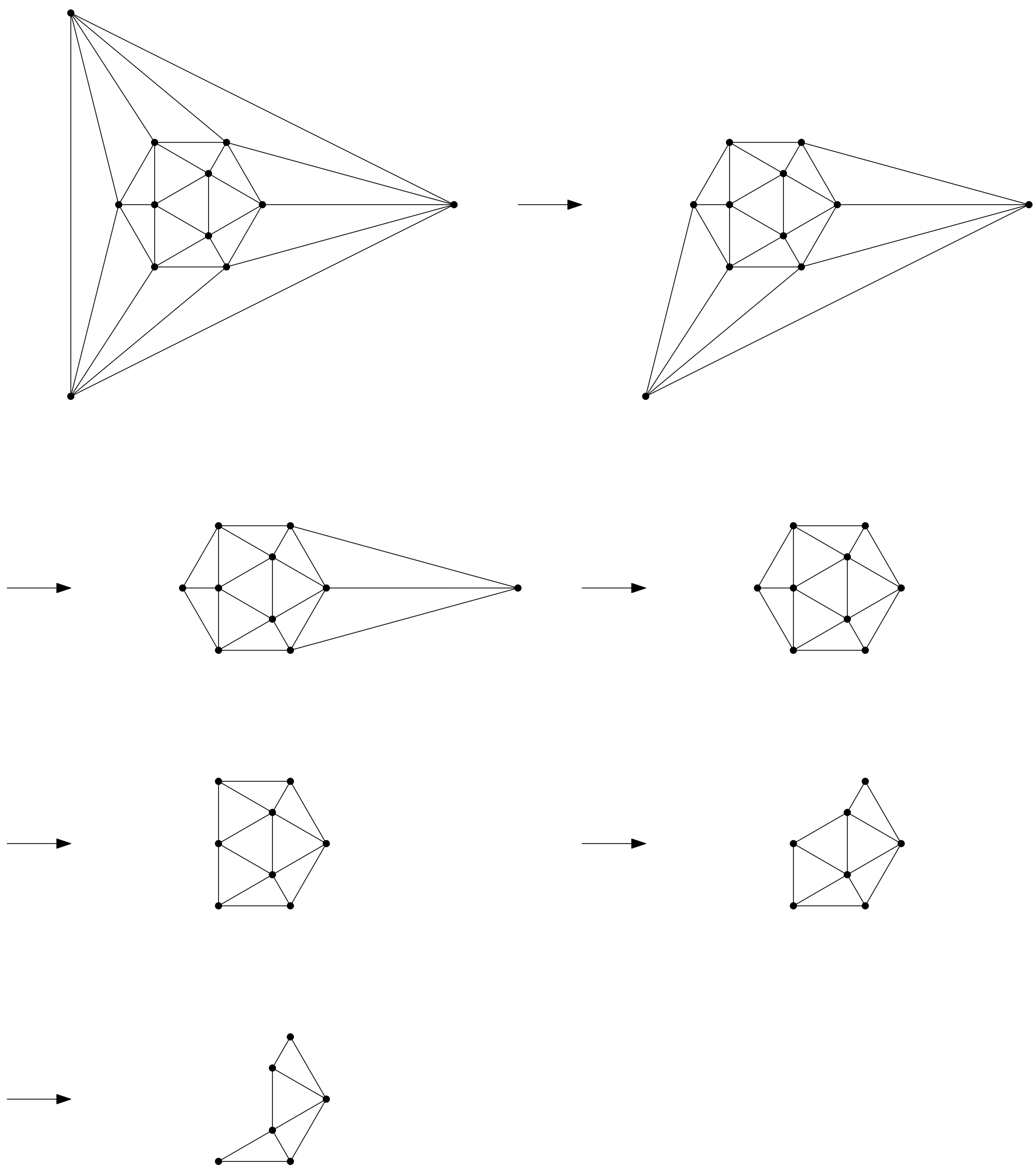
5. The proof gave us a *recursive* algorithm.

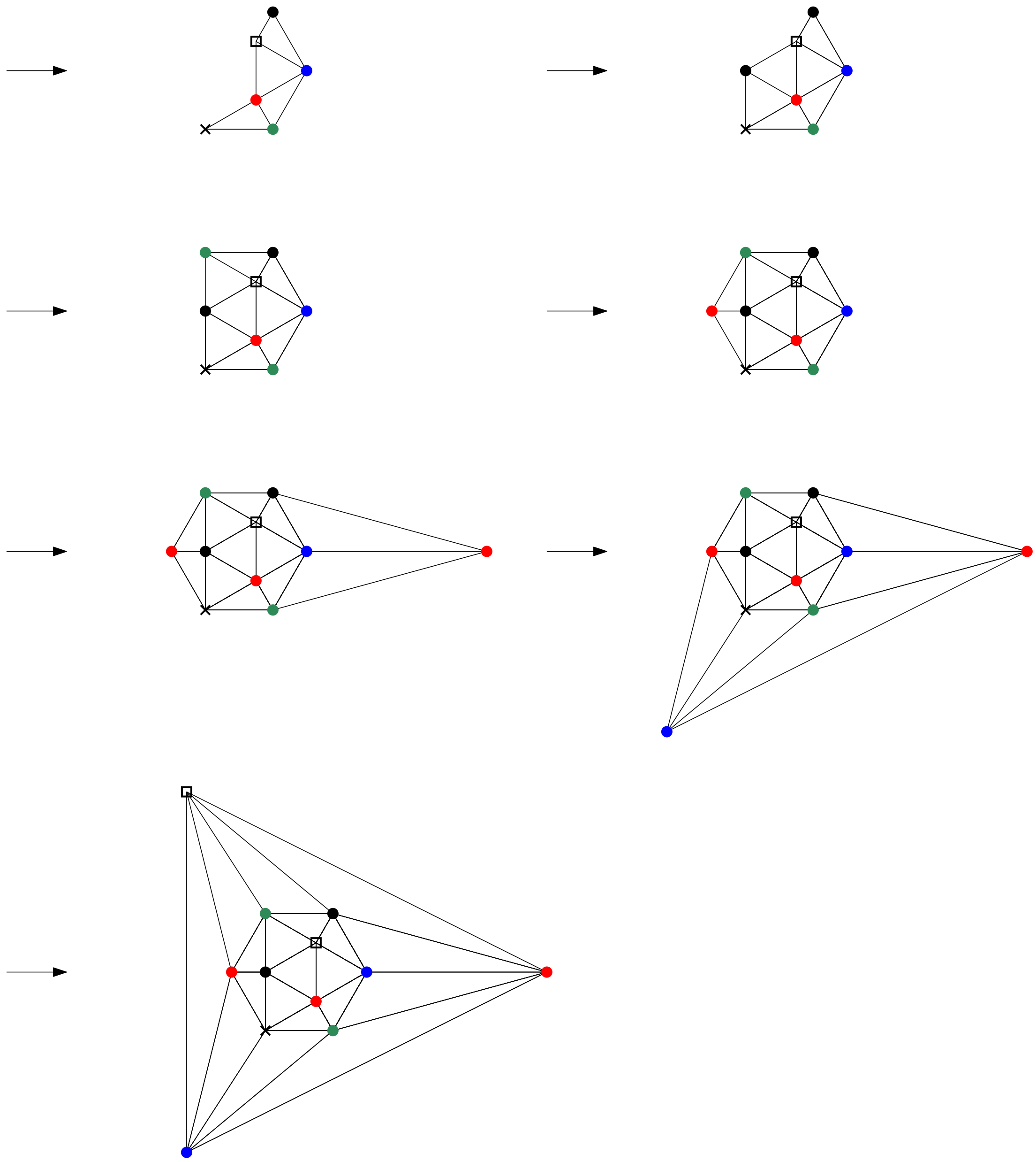
   Let $G = (V, E)$ be the input graph.

If G has at most 6 vertices, then we give a different color to each vertex.

If G has more than 6 vertices, we find one vertex $u \in V$ of degree at most 5 (such a vertex exists by a result proved in class). We erase u as well as all edges adjacent to it. We obtain a new graph G' that contains one fewer vertex than G. We 6-color G' recursively. Since u is of degree at most 5, u has at most 5 neighbours. Therefore, we can give u a color that is different to all its neighbours.

7.





9. C_n is defined for all $n \geq 3$. For each $n \geq 3$, C_n is a cycle. Therefore, we can draw it without any intersecting edges.

   We conclude C_n is planar for all $n \geq 3$.

11.

   (a)

$n = 1$

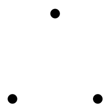
$G_{1,1}$

$n = 2$

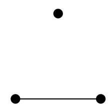
$G_{2,1}$

$G_{2,2}$

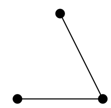
$n = 3$



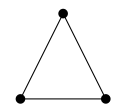
$G_{3,1}$



$G_{3,2}$

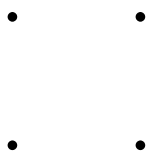


$G_{3,3}$

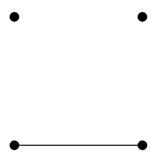


$G_{3,4}$

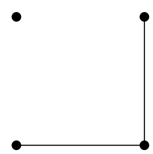
$n = 4$



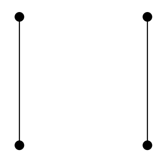
$G_{4,1}$



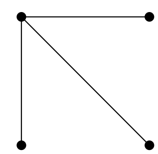
$G_{4,2}$



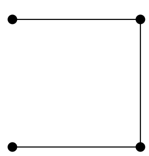
$G_{4,3}$



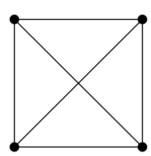
$G_{4,4}$



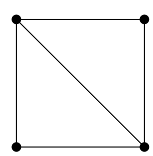
$G_{4,5}$



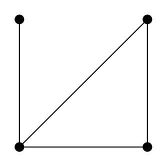
$G_{4,6}$



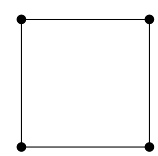
$G_{4,7}$



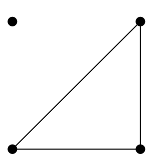
$G_{4,8}$



$G_{4,9}$

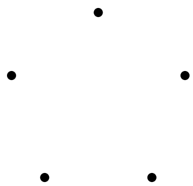


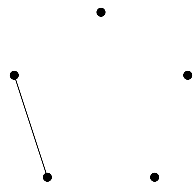
$G_{4,10}$

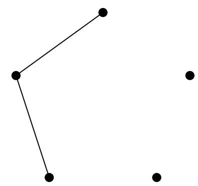


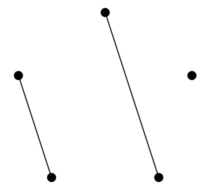
$G_{4,11}$

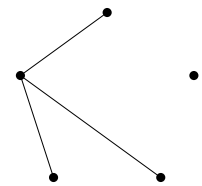
$n = 5$

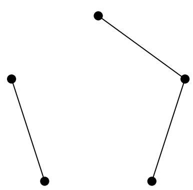

$G_{5,1}$

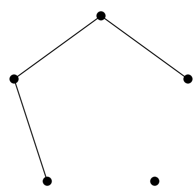

$G_{5,2}$

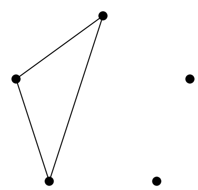

$G_{5,3}$

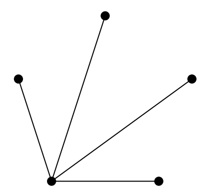

$G_{5,4}$

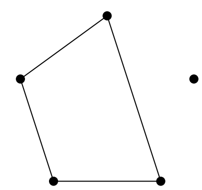

$G_{5,5}$

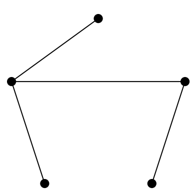

$G_{5,6}$

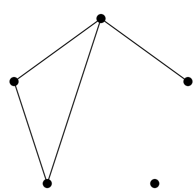

$G_{5,7}$

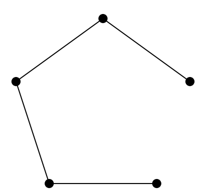

$G_{5,8}$

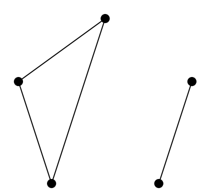

$G_{5,9}$

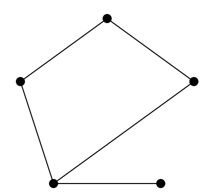

$G_{5,10}$

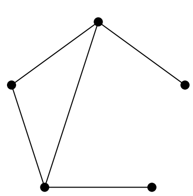

$G_{5,11}$

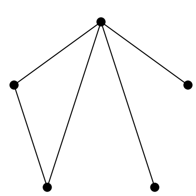

$G_{5,12}$

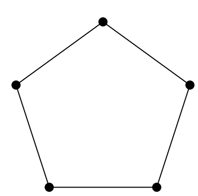

$G_{5,13}$

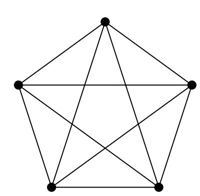

$G_{5,14}$

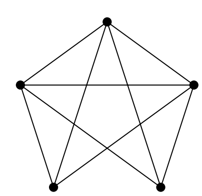

$G_{5,15}$

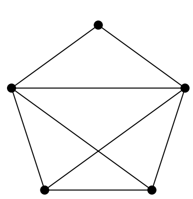

$G_{5,16}$

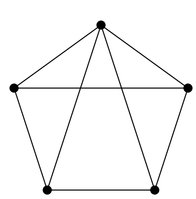

$G_{5,17}$

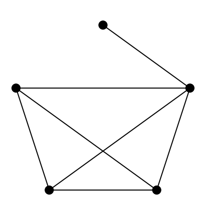

$G_{5,18}$

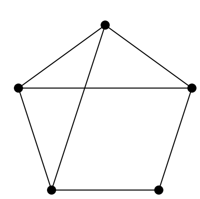

$G_{5,19}$

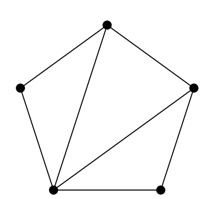

$G_{5,20}$

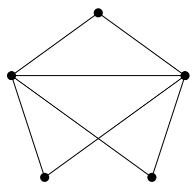

$G_{5,21}$


$G_{5,22}$

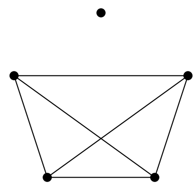

$G_{5,23}$


$G_{5,24}$

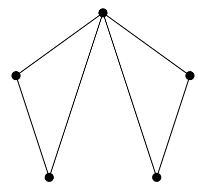

$G_{5,25}$



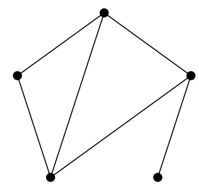
$G_{5,26}$



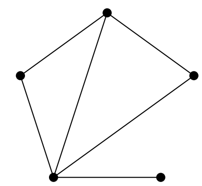
$G_{5,27}$



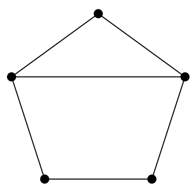
$G_{5,28}$



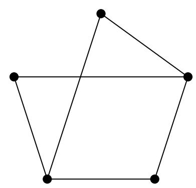
$G_{5,29}$



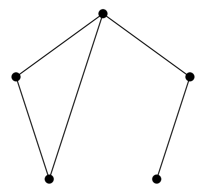
$G_{5,30}$



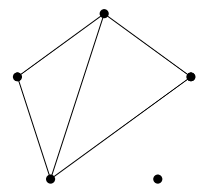
$G_{5,31}$



$G_{5,32}$



$G_{5,33}$



$G_{5,34}$

(b) All the previous graphs except $G_{5,19}$

(c) $G_{1,1}$, $G_{2,2}$, $G_{3,3}$, $G_{3,4}$, $G_{4,5}$, $G_{4,6}$, $G_{4,7}$, $G_{4,8}$, $G_{4,9}$, $G_{4,10}$, $G_{5,9}$, $G_{5,11}$, $G_{5,13}$, $G_{5,15}$, $G_{5,16}$, $G_{5,17}$, $G_{5,18}$, $G_{5,19}$, $G_{5,20}$, $G_{5,21}$, $G_{5,22}$, $G_{5,23}$, $G_{5,24}$, $G_{5,25}$, $G_{5,26}$, $G_{5,28}$, $G_{5,29}$, $G_{5,30}$, $G_{5,31}$, $G_{5,32}$ and $G_{5,33}$.

(d) $G_{1,1}$, $G_{2,2}$, $G_{3,3}$, $G_{3,4}$, $G_{4,5}$, $G_{4,6}$, $G_{4,7}$, $G_{4,8}$, $G_{4,9}$, $G_{4,10}$, $G_{5,9}$, $G_{5,11}$, $G_{5,13}$, $G_{5,15}$, $G_{5,16}$, $G_{5,17}$, $G_{5,18}$, $G_{5,20}$, $G_{5,21}$, $G_{5,22}$, $G_{5,23}$, $G_{5,24}$, $G_{5,25}$, $G_{5,26}$, $G_{5,28}$, $G_{5,29}$, $G_{5,30}$, $G_{5,31}$, $G_{5,32}$ and $G_{5,33}$.

13. **If n is even**, then the chromatic number for C_n is 2. To show this, assume n is even. Number the vertices : $V = \{v_0, v_1, ..., v_{n-1}\}$ (we know that $n-1$ is odd). Each vertex with even index is given the color *red* those with odd index are colored *blue*. For all even index vertices, its two neighbours have odd indices. Likewise, for all odd index vertices, its two neighbours have even indices. Therefore we can use 2-coloration.

If n is odd, then the chromatic number for C_n is 3. The explanation is a little longer in this case. Assume n is odd. Number the vertices : $V = \{v_0, v_1, ..., v_{n-1}\}$ (we know that $n-1$ is even). Each vertex with even index ***except*** v_{n-1} is given the color *red*. Each vertex with odd index is given the color *blue*. The only remaining vertex is v_{n-1} which we assign the color *green*.

The vertices v_0 and v_{n-1} have a special property : they have one even index neighbour and one odd index neighbour. For all the other vertices v_i, if v_i is of even index, its two neighbours have odd indices; and if v_i is of odd index, its two neighbours have even indices. This is a 3-coloration.

If n is odd, we can't color it with 2 colors because in that case, C_n is not bipartite. Let's prove this by contradiction. Suppose that n is odd and C_n is bipartite. We need to find two sets V_1 and V_2 such that $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \{\ \}$. Since the graph is a cycle, for all $0 \leq i \leq n-1$, v_i and v_{i+1} cannot be in the same set. We have no choice but to sort : $v_0, v_2, ..., v_{n-3} \in V_1$ et $v_1, v_3, ..., v_{n-2} \in V_2$. But then, there is nowhere to place v_{n-1} since v_{n-1} is adjacent to both v_0 and v_{n-2}. This is a contradiction.

15. We saw in class that the chromatic number of a graph G is 2 if and only if G is bipartite. By definition, $K_{m,n}$ is bipartite. Therefore the chromatic number of $K_{m,n}$ is 2.

17. For all $n \geq 3$, the graph C_n is a cycle ! In following this cycle, we travel over every edge exactly once. So for all $n \geq 3$, C_n contains an Eulerian cycle.

19. We saw the following result in class : a graph contains an Eulerian path (that is not a cycle) if and only if it contains exactly two vertices of odd degree.

    If $n = 1$, it's a special case : K_1 contains just a single vertex. According to the definitions given in class, this corresponds to a path of length 0. So K_1 contains an Eulerian path that is not a cycle.

    If $n = 2$, K_2 is a path that is not a cycle and travels over every edge. So K_2 contains an Eulerian path that is not a cycle.

    If $n \geq 3$ and n is even, then we do not satisfy the condition because though all the vertices are of odd degree, there are more than two vertices. So K_n does not contain an Eulerian path.

    If $n \geq 3$ and n is odd, then we do not satisfy the condition because all the vertices are of even degree. So K_n does not contain an Eulerian path.

21. We saw the following result in class : a graph contains an Eulerian path (that is not a cycle) if and only if it contains exactly two vertices of odd degree.

    The notation $K_{m,n}$ signifies that in the definition of a bipartite graph, $|V_1| = m$ and $|V_2| = n$. Since we need exactly two vertices of odd degree, we take

    $$m = 1 \quad \text{and} \quad n = 1$$

    or

    $$m = 1 \quad \text{and} \quad n = 2$$

    or

    $$m = 2 \quad \text{and} \quad n \geq 3 \text{ odd.}$$

23. For all $n \geq 3$, the graph C_n is a cycle ! So in following it, we visit all vertices exactly once. Therefore, for all $n \geq 3$, C_n contains a Hamiltonian cycle.

25. For all $n \geq 1$, K_n contains a Hamiltonian cycle. Consider the set of vertices $V = \{v_1, v_2, ..., v_n\}$. Start at v_1. Since it is a complete graph, we can find an edge leading to v_2. Since it is a complete graph, we can find an edge leading to v_3, and so on until we reach v_n. Therefore, we have a path that visits each vertex exactly once. Furthermore, this path is not a cycle because we do not return to v_1.

27. **Suppose that** $n = m$. Let $V_1 = \{v_1, v_2, ..., v_n\}$ and $V_2 = \{u_1, u_2, ..., u_n\}$. Starting from v_1, we can follow the path given by the sequence :

    $$v_1, u_1, v_2, u_2, v_3, u_3, ..., v_{n-1}, u_{n-1}, v_n, u_n.$$

So $K_{m,n}$ contains a Hamiltonian path that is not a cycle.

Suppose that $n = m + 1$. Let $V_1 = \{v_1, v_2, ..., v_{n-1}, v_n\}$ and $V_2 = \{u_1, u_2, ..., u_{n-1}\}$. Starting from v_1, we can follow that path given by the sequence :

$$v_1, u_1, v_2, u_2, v_3, u_3, ..., v_{n-1}, u_{n-1}, v_n.$$

So $K_{m,n}$ contains a Hamiltonian path that is not a cycle.

Suppose that $n \geq m + 2$. Then $K_{m,n}$ does not contain a Hamiltonian path. Here's why. Since all the edges of $K_{m,n}$ are those connecting V_1 and V_2, a Hamiltonian path needs to alternate between V_1 et V_2. Suppose that $|V_1| = m$ et $|V_2| = n$. If we start in V_1, since we need to alternate between V_1 and V_2, we can only visit m vertices in V_2 (but $m < n$). If we start in V_2, since we need to alternate between V_2 and V_1, we can only visit $m + 1$ vertices in V_2 (but $m + 1 < m + 2 \leq n$). So in all cases, we do not reach all the vertices in V_2.

29. For all $n \geq 1$, find the size of a maximum matching in K_n.

Suppose that n **is even**. Let $V = \{v_0, v_1, ..., v_{n-1}\}$ (so $n - 1$ is odd). Since there exists an edge for each pair of vertices, we can create the following matching

$$\big\{\{v_0, v_1\}, \{v_2, v_3\}, ..., \{v_{n-4}, v_{n-3}\}, \{v_{n-2}, v_{n-1}\}\big\}$$

where all the vertices have been listed. Therefore we cannot add any additional edges. We conclude that the size of a maximum matching is $\frac{n}{2}$.

Suppose that n **is odd**. Let $V = \{v_0, v_1, ..., v_{n-1}\}$ (so $n - 1$ is even). Since each edge connects two vertices and a vertex cannot be repeated in a matching, we cannot find a matching of size greater than $\frac{n-1}{2}$ (upper bound). Actually, a matching of this size exists. We choose

$$\big\{\{v_0, v_1\}, \{v_2, v_3\}, ..., \{v_{n-5}, v_{n-4}\}, \{v_{n-3}, v_{n-2}\}\big\}.$$

We conclude that the size of a maximum matching is $\frac{n-1}{2}$.

Note : We could combine both cases by saying that in K_n, the size of a maximum matching is $\lfloor \frac{n}{2} \rfloor$, but this is not necessary.