COMP 2804 — Assignment 1

Due: Sunday September 27, 11:55 pm.

Assignment Policy:

- Your assignment must be submitted as one single PDF file through cuLearn.
- **Late assignments will not be accepted. I will not reply to emails of the type "my internet connection broke down at 11:53pm" or "my scanner stopped working at 11:54pm", or "my dog ate my laptop charger".**
- You are encouraged to collaborate on assignments, but at the level of discussion only. When writing your solutions, you must do so in your own words.
- Past experience has shown conclusively that those who do not put adequate effort into the assignments do not learn the material and have a probability near 1 of doing poorly on the exams.
- When writing your solutions, you must follow the guidelines below.
  - You must justify your answers.
  - The answers should be concise, clear and neat.
  - When presenting proofs, every step should be justified.

Question 1:

- Write your name and student number.

Question 2: Recall that K_n, where $n \geq 2$ is an integer, denotes the *complete graph* with n vertices. This graph has $\binom{n}{2}$ edges, i.e., one edge for each pair of distinct vertices.

Let $k \geq 1$ be an integer and let L be a set of size k. If each edge of K_n is labeled with one element of L, then we say that K_n is *k-labeled*. Three pairwise distinct vertices u, v, and w of K_n define a *boring* triangle, if the three edges uv, uw, and vw have the same label.

In this question, you will prove the following statement $S(n, k)$ for all integers n and k with $n \geq 3 \cdot k!$:

$$S(n, k)\text{: Every } k\text{-labeled } K_n \text{ has a boring triangle.}$$

(Note that in class, we have proved statement $S(n, 2)$ for all $n \geq 6$.) The proof will be by induction on k.

- Base case: Explain, in a few sentences, why statement $S(n, 1)$ is true for all integers $n \geq 3$.

- Induction step: Let $k \geq 2$ and assume that statement $S(m, k-1)$ is true for all integers $m \geq 3 \cdot (k-1)!$. Prove that statement $S(n, k)$ is true for all integers $n \geq 3 \cdot k!$.

  Hint: Let u be an arbitrary vertex of K_n. Argue that among all edges that have u as a vertex, at least $\lceil (n-1)/k \rceil$ have the same label.

Question 3: Let $k \geq 2$ be an integer and consider the set $S = \{1, 2, \ldots, k\}$. For any integer $n \geq 2$, determine the number of sequences $(a_1, a_2, \ldots, a_n)$ of length n, where each element belongs to S and no two consecutive elements are the same.

Question 4: Zoltan's House of Pizza is a popular restaurant in downtown Ottawa. The daily dinner special is Zoltan's Meal Deal: You choose between standard dough and gluten free dough; you choose one of chicken, beef, pork, and shrimp; you choose one out of five different sauces; you choose three out of seven different cheeses; you choose any subset of twenty different toppings; and, finally, you choose four out of nine different beers.

- Determine the number of Zoltan's Meal Deals, if the cheeses chosen must be distinct and the beers chosen must be distinct.
- Determine the number of Zoltan's Meal Deals, if the cheeses chosen are not necessarily distinct and the beers chosen are not necessarily distinct.

Question 5: This fall term, 230 students have registered for COMP 2804B. While watching video lectures, 150 students do not wear pants and 110 students drink beer.

Determine the best possible lower bound on the number of students who do not wear pants and drink beer while watching video lectures.

Question 6: In class, we have seen how to determine the number of solutions of the equation

$$x_1 + x_2 + x_3 = 99, \tag{1}$$

where $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$ are integers.

- Determine the number of solutions of (1), where $x_1 \geq 40$, $x_2 \geq 0$, and $x_3 \geq 0$ are integers.
- Determine the number of solutions of (1), where $x_1 \geq 0$, $x_2 \geq 50$, and $x_3 \geq 0$ are integers.
- Determine the number of solutions of (1), where $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 55$ are integers.
- Determine the number of solutions of (1), where $x_1 \geq 40$, $x_2 \geq 50$, and $x_3 \geq 0$ are integers.

- Use the Principle of Inclusion and Exclusion to determine the number of solutions of (1), where $0 \leq x_1 \leq 39$, $0 \leq x_2 \leq 49$, and $0 \leq x_3 \leq 54$ are integers.

Question 7: Let $n \geq 2$ be an integer and consider the set $S = \{1, 2, \ldots, n\}$. A permutation of S can be regarded as a bijection $f : S \to S$. For any integer $i \geq 1$, define

$$f^i(x) = \underbrace{f(f(f(\cdots f}_{i \text{ times}}(x)\cdots))).$$

In words, to obtain $f^i(x)$, we start with x and apply the function i times.

If x is an element of S, then the *cycle* of x is the sequence

$$(x, f^1(x), f^2(x), \ldots, f^{m-1}(x)),$$

where m is the smallest positive integer such that $f^m(x) = x$.

The *cycle decomposition* of f is obtained as follows:

Step 1: Initially, all elements of S are unmarked and the cycle decomposition is empty.

Step 2: Repeat the following, until all elements of S have been marked: Take the smallest unmarked element, say x, in S. Mark all elements in the cycle of x, and add this cycle to the cycle decomposition.

For example, the cycle decomposition of the permutation

$f(x)$	7	4	6	8	1	3	5	2
x	1	2	3	4	5	6	7	8

is given by

$$(1, 7, 5), (2, 4, 8), (3, 6).$$

This cycle decompositions consists of three cycles, having lengths 3, 3, and 2, respectively.

- Let x be an element of S. Explain why there exists a positive integer m such that $f^m(x) = x$.
- Prove or disprove the following statement: Let k be an integer with $k > n/2$. The cycle decomposition can have more than one cycle of length k.
- Let k be an integer with $k > n/2$. Prove that the number of permutations of S whose cycle decompositions contain a cycle of length k is equal to

$$\binom{n}{k} \cdot (k-1)! \cdot (n-k)!. \tag{2}$$

- Use algebra to simplify the expression in (2) to $n!/k$.

Question 8: Let $n \geq 2$ be an integer and consider the set $S = \{1, 2, \ldots, n\}$. Recall that a permutation of S is an ordered sequence $a_1, a_2, \ldots, a_n$, in which every element of S occurs exactly once.

Let i be an integer with $1 \leq i \leq n$. We say that i is a *cool index*, if $a_i \neq i$. We say that i is a *super cool index*, if i is cool and none of the indices $i+1, i+2, \ldots, n$ is cool; in other words, i is the rightmost cool index in the permutation.

For example, in the permutation below, the indices 1 and 4 are cool, whereas the index 4 is super cool.

a_i	4	2	3	1	5	6	7	8
i	1	2	3	4	5	6	7	8

- Determine the number of permutations of the set S that have at least one cool index.
- Prove or disprove the following statement: There exists a permutation of S in which the index 1 is super cool.
- Let k be an integer with $1 \leq k \leq n-1$. Determine the number of permutations of S in which the index $k+1$ is super cool.
- Use the above results to prove that

$$\sum_{k=1}^{n-1} k \cdot k! = n! - 1.$$

Question 9: Let $n \geq 1$ be an integer and consider the set $S = \{0, 1, \ldots, 3n\}$. For any integer k, let N_k be the number of subsets X of S for which $|X| = 2n+1$ and $\min(X) = k$.

- Determine the values of k for which $N_k \neq 0$.
- Let k be an integer with $0 \leq k \leq n$. Determine the value of N_k.
- Use the above results to prove that

$$\sum_{k=0}^{n} \binom{3n-k}{2n} = \binom{3n+1}{n}.$$