COMP 3803 — Solutions Assignment 2

Question 1: Write your name and student number.

Solution: Karim Benzema, 9

Question 2: Consider the language A consisting of all strings over the alphabet $\{a, b\}$ that contain both aba and bab as substrings. Give a regular expression that describes the language A. As always, justify your answer.

Solution: Each string in the language A is of one of the following four types:

- It contains the string $abab$.

  These start with a string in $\{a, b\}^*$, followed by $abab$, and end with a string in $\{a, b\}^*$.
- It contains the string $baba$.

  These start with a string in $\{a, b\}^*$, followed by $baba$, and end with a string in $\{a, b\}^*$.
- aba is to the left of bab and they are not overlapping.

  These start with a string in $\{a, b\}^*$, followed by aba, then a string in $\{a, b\}^*$, then bab, and end with a string in $\{a, b\}^*$.
- bab is to the left of aba and they are not overlapping.

  These start with a string in $\{a, b\}^*$, followed by bab, then a string in $\{a, b\}^*$, then aba, and end with a string in $\{a, b\}^*$.

This gives the regular expression

$$(a \cup b)^* abab (a \cup b)^* \cup (a \cup b)^* baba (a \cup b)^* \cup (a \cup b)^* aba (a \cup b)^* bab (a \cup b)^* \cup (a \cup b)^* bab (a \cup b)^* aba (a \cup b)^*$$

Question 3: Let A be the language over the alphabet $\{a, b\}$ that is described by the regular expression aa. Give a regular expression that describes the complement $\overline{A}$ of A. As always, justify your answer.

Solution: The language described by the regular expression aa is $A = \{aa\}$. We need a regular expression for its complement, i.e., all strings that are not equal to aa.

Each string in $\overline{A}$ is of one of the following three types:

- Any string of length at most one.
- It has length exactly two and is not equal to aa.
- It has length at least three.

This gives the regular expression

$$\varepsilon \cup a \cup b \cup ab \cup ba \cup bb \cup (a \cup b)(a \cup b)(a \cup b)(a \cup b)^*$$

Question 4: In this question, the alphabet is $\{a, b\}$. A *block* in a string is a maximal substring all of whose symbols are the same. For example, the string *aaabbaa* has three blocks: *aaa*, *bb*, and *aa*.

Let A be the language of all strings w such that every block in w has length two or three. The empty string is in A, as is the string *aaabbaa*.

Give a regular expression that describes the language A. As always, justify your answer.

Solution: *Throughout the solution, a block always refers to a block of length two or three.*

Each string in the language A is of one of the following two types:

- The string starts with an a-block, and the a-blocks and b-blocks alternate. The total number of blocks can be even or odd.
- The string starts with a b-block, and the b-blocks and a-blocks alternate. The total number of blocks can be even or odd.

This gives the regular expression

$$((aa \cup aaa)(bb \cup bbb))^*(\varepsilon \cup aa \cup aaa) \cup ((bb \cup bbb)(aa \cup aaa))^*(\varepsilon \cup bb \cup bbb)$$

Note that ε is in the language described by this regular expression.

Question 5: Use the construction given in class to convert the regular expression

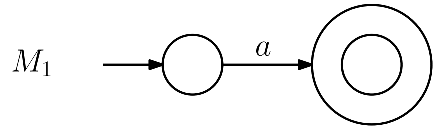
$$a \cup ba^*$$

to an NFA. Do not simplify your NFA; just apply the construction rules "without thinking".

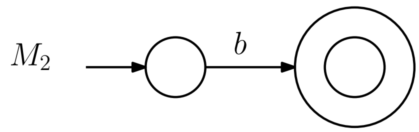
Solution: We first consider how the regular expression is "built":

- Take the regular expression a.
- Take the regular expression b.
- Take the regular expression a, and turn it into the regular expression a^*.
- Take the regular expressions b and a^*, and combine them into the regular expression ba^*.
- Take the regular expressions a and ba^*, and combine them into the regular expression $a \cup ba^*$.

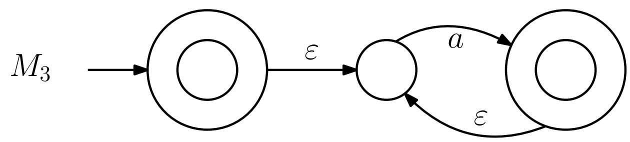
First, we construct an NFA M_1 that accepts the language described by the regular expression a:



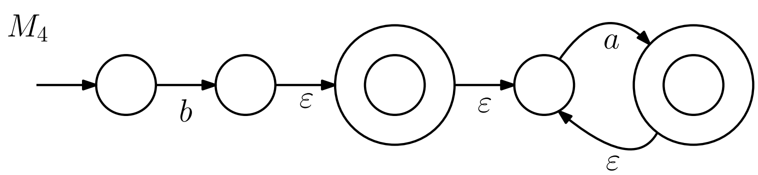
Next, we construct an NFA M_2 that accepts the language described by the regular expression b:



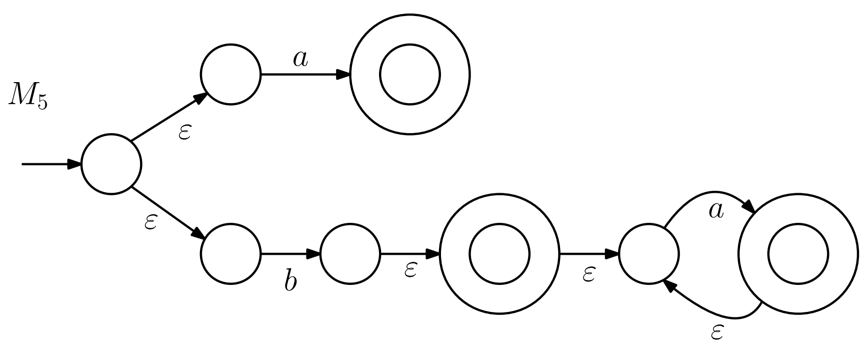
Next, we apply the star construction to M_1. This gives an NFA M_3 that accepts the language described by the regular expression a^*:



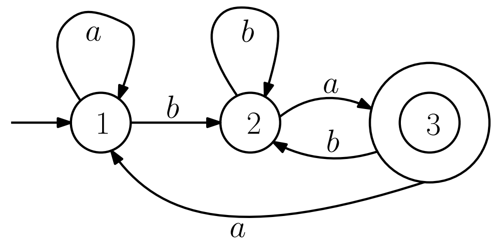
Next, we apply the concatenate construction to M_2 and M_3. This gives an NFA M_4 that accepts the language described by the regular expression ba^*:



Finally, we apply the union construction to M_1 and M_4. This gives an NFA M_5 that accepts the language described by the regular expression $a \cup ba^*$:



Question 6: Use the construction given in class to convert the following DFA to a regular expression.



Solution: For each state $i = 1, 2, 3$, we define L_i to be the set of all strings w in $\{a, b\}^*$ such that the path in the state diagram that starts in state i and corresponds to w ends in the accept state. We obtain the following set of equations:

$$L_1 = aL_1 \cup bL_2 \quad (1)$$
$$L_2 = aL_3 \cup bL_2 \quad (2)$$
$$L_3 = \epsilon \cup aL_1 \cup bL_2 \quad (3)$$

Since 1 is the start state, we need a regular expression for L_1.

We use the following tool to solve these equations:

$$\text{If } L = BL \cup C \text{ and } \epsilon \notin B, \text{ then } L = B^*C. \quad (4)$$

We solve the equations (1), (2), and (3), in the following way: By substituting (3) into (2), we obtain

$$L_2 = a(\epsilon \cup aL_1 \cup bL_2) \cup bL_2,$$

which we rewrite as

$$L_2 = (ab \cup b)L_2 \cup (a \cup aaL_1). \quad (5)$$

This equation is in the form of (4), with $L = L_2$, $B = (ab \cup b)$, and $C = (a \cup aaL_1)$. Since ϵ is not in the language described by B, we can apply (4) to (5), and we obtain

$$L_2 = (ab \cup b)^*(a \cup aaL_1),$$

which we rewrite as

$$L_2 = (ab \cup b)^*a \cup (ab \cup b)^*aaL_1. \quad (6)$$

By substituting (6) into (1), we obtain

$$L_1 = aL_1 \cup b\left((ab \cup b)^*a \cup (ab \cup b)^*aaL_1\right),$$

which we rewrite as

$$L_1 = \left(a \cup b(ab \cup b)^*aa\right) L_1 \cup b(ab \cup b)^*a. \quad (7)$$

This equation is in the form of (4), with $L = L_1$,

$$B = (a \cup b(ab \cup b)^*aa)$$

and $C = b(ab \cup b)^*a$. Since ϵ is not in the language described by B, we can apply (4) to (7), and we obtain

$$L_1 = (a \cup b(ab \cup b)^*aa)^*\, b(ab \cup b)^*a.$$

Question 7: Use the Pumping Lemma to prove that the following languages are not regular. In all cases, the alphabet is $\{a, b\}$.

1. $\{a^n u : n \geq 0, u \in \{a,b\}^*, |u| \leq n\}$.
2. $\{a^m b^n : m \geq 0, n \geq 0,\ n \text{ is a multiple of } m\}$.
3. $\{a^m b^n : m \geq 0, n \geq 0,\ m \text{ is a multiple of } n\}$.
   (a) Professor Justin Bieber claims that this can be proven by taking the string $s = a^p b^p$, where p is the Pumping length. Show that Professor Bieber is (again!) wrong.
   (b) Now give a correct proof.
4. $\{uv : u \in \{a,b\}^*, v \in \{a,b\}^*, u = v^R\}$.
   Note: If $v = v_1 v_2 \cdots v_n$ is a string, then $v^R = v_n v_{n-1} \cdots v_1$ is the reverse of v.

Solution: First, we do

$$A = \{a^n u : n \geq 0, u \in \{a,b\}^*, |u| \leq n\}.$$

Assume the language A is regular. Let $p \geq 1$ be the pumping length, as given by the Pumping Lemma. Let $s = a^p b^p$. We can write $s = a^p u$, where $u = b^p$. Since $|u| \leq p$, the string s is in A. Also, $|s| = 2p \geq p$. Hence, by the Pumping Lemma, we can write $s = xyz$, where

1. $y \neq \epsilon$,
2. $|xy| \leq p$, and
3. $xy^i z \in A$, for all $i \geq 0$.

Since $|xy| \leq p$, the string y contains only as. Since $y \neq \epsilon$, the string y contains at least one a. Let $k = |y|$, so that $y = a^k$. Note that $1 \leq k \leq p$. By the Pumping Lemma, the string

$$s' = xy^0 z = a^{p-k} b^p$$

is in A. Hence, there is a j such that $s' = a^j u$, where $u \in \{a,b\}^*$ and $|u| \leq j$.

However, since $j \leq k - p$, we must have $|u| \geq p > p - k \geq j$. Thus, s' is not in A. So we have a contradiction, and we can conclude that A is not regular.

Next we do

$$B = \{a^m b^n : m \geq 0, n \geq 0, \ n \text{ is a multiple of } m\}.$$

Assume the language B is regular. Let $p \geq 1$ be the pumping length, as given by the Pumping Lemma. Let $s = a^p b^p$. Since p is a multiple of p, the string s is in B. Also, $|s| = 2p \geq p$. Hence, by the Pumping Lemma, we can write $s = xyz$, where

1. $y \neq \epsilon$,
2. $|xy| \leq p$, and
3. $xy^i z \in B$, for all $i \geq 0$.

Since $y \neq \epsilon$ and $|xy| \leq p$, the string y has the form $y = a^k$, for some integer k with $1 \leq k \leq p$. Consider the string

$$s' = xy^2 z = xyyz = a^{p+k} b^p.$$

By the Pumping Lemma, s' is in B. However, since p is not a multiple of $p + k$, the string s' is not in B. So we have a contradiction, and we can conclude that B is not regular.

Next we do

$$C = \{a^m b^n : m \geq 0, n \geq 0, \ m \text{ is a multiple of } n\}.$$

We first show why Professor Bieber should fail COMP 3803: The string $s = a^p b^p$ is in C because p is a multiple of p. Also $|s| = 2p \geq p$. Hence, by the Pumping Lemma, we can write $s = xyz$, where

1. $y \neq \epsilon$,
2. $|xy| \leq p$, and
3. $xy^i z \in C$, for all $i \geq 0$.

Since $y \neq \epsilon$ and $|xy| \leq p$, the string y has the form $y = a^k$, for some integer k with $1 \leq k \leq p$.

It may happen that $k = p$, i.e., $x = \varepsilon$ and $y = a^p$. In this case, for every $i \geq 0$,

$$xy^i z = a^{ip} b^p.$$

Since ip is a multiple of p, all these strings $xy^i z$ are in C. In other words, we do not get a contradiction.

Here are two correct proofs.

First proof: Assume the language C is regular. The reverse language C^R, obtained by reversing all strings in C is equal to

$$C^R = \{b^n a^m : m \geq 0, n \geq 0, \ m \text{ is a multiple of } n\}.$$

We have seen in class that the reverse of a regular language is also regular. Thus, C^R is regular. But, by swapping a and b, C^R is equal to B, the language in part 2. Thus, B is regular. This is a contradiction. We conclude that C is not regular.

Second proof: Assume the language C is regular. Let $p \geq 1$ be the pumping length, as given by the Pumping Lemma. Let $s = a^{p+1}b^{p+1}$. Since $p+1$ is a multiple of $p+1$, the string s is in B. Also, $|s| = 2p+2 \geq p$. Hence, by the Pumping Lemma, we can write $s = xyz$, where

1. $y \neq \epsilon$,
2. $|xy| \leq p$, and
3. $xy^iz \in C$, for all $i \geq 0$.

Since $y \neq \epsilon$ and $|xy| \leq p$, the string y has the form $y = a^k$, for some integer k with $1 \leq k \leq p$. Consider the string

$$s' = xy^2z = xyyz = a^{p+1+k}b^{p+1}.$$

By the Pumping Lemma, s' is in C. Thus, $p+1+k$ is a multiple of $p+1$. However,

$$p+1 < p+1+k \leq p+1+p < 2(p+1),$$

i.e., $p+1+k$ is strictly between two consecutive multiples of $p+1$. It follows that $p+1+k$ is not a multiple of $p+1$. So we have a contradiction, and we can conclude that C is not regular.

Finally, we do

$$D = \{uv : u \in \{a,b\}^*, v \in \{a,b\}^*, u = v^R\}.$$

Assume the language D is regular. Let $p \geq 1$ be the pumping length, as given by the Pumping Lemma. Let $s = a^pbpa^p$. since the first half is the reverse of the second half, the string s is in D. Also, $|s| = 2p+2 \geq p$. Hence, by the Pumping Lemma, we can write $s = xyz$, where

1. $y \neq \epsilon$,
2. $|xy| \leq p$, and
3. $xy^iz \in D$, for all $i \geq 0$.

Since $|xy| \leq p$, the string y has the form $y = a^k$, for some integer k with $1 \leq k \leq p$. consider the string

$$s' = xy^2z = xyyz = a^{p+k}bba^p.$$

By the Pumping Lemma, s' is in D. Note that the reverse of any string in D is also in D. It is clear, however, that the reverse of s' is not in D. This is a contradiction. So we have a contradiction, and we can conclude that D is not regular.

Question 8: Consider the language

$$A = \{a^{n^2} : n \geq 0\} \cup \{a^{2n+1} : n \geq 0\};$$

note that the alphabet is $\{a\}$.

In this question, you will prove that A is not regular, but the concatenation AA is regular.

1. Prove that A is not a regular language.

   Hint: IGNORE THIS HINT!!!!!!!! What is

$$A \cap \{a^{2n+1} : n \geq 0\}?$$

2. Prove that

$$AA = \{a^n : n \geq 0\}.$$

3. Prove that AA is a regular language.

Solution: SINCE THE HINT IS A PoS, THIS QUESTION WILL NOT BE MARKED.

We start with part 1. Assume the language A is regular. Let $p \geq 1$ be the pumping length, as given by the Pumping Lemma. Let $s = a^{(2p)^2}$, i.e., the string consisting of $(2p)^2 = 4p^2$ many a's. Then s is a string in A and $|s| = 4p^2 \geq p$. Hence, by the Pumping Lemma, we can write $s = xyz$, where

1. $y \neq \epsilon$,
2. $|xy| \leq p$, and
3. $xy^i z \in A$, for all $i \geq 0$.

Since $y \neq \epsilon$ and $|xy| \leq p$, the string y has the form $y = a^k$, for some integer k with $1 \leq k \leq p$. Consider the string

$$s' = xy^3z = xyyyz = a^{(2p)^2+2k}.$$

By the Pumping Lemma, s' is in A. We now argue that s' is not in A. First, the length of s' is even. Second

$$(2p)^2 < (2p)^2 + 2k \leq (2p)^2 + 2p < (2p+1)^2$$

and, thus, the length of s' is not a square (because it is strictly between two consecutive squares). So we have a contradiction, and we can conclude that A is not regular.

For part 2., it is clear that $AA \subseteq \{a^n : n \geq 0\}$. We show that $\{a^n : n \geq 0\} \subseteq AA$:

- Since $\varepsilon \in A$, the string $a^0 = \varepsilon\varepsilon$ is in AA.
- Let $n \geq 0$. Since both ε and a^{2n+1} are in A, the string $a^{2n+1} = \varepsilon a^{2n+1}$ is in AA.
- Let $n \geq 2$ be an even integer. Then $n = 1 + (n-1)$, which is a sum of two odd integers. Since both a and a^{n-1} are in A, the string $a^n = aa^{n-1}$ is in AA.

For part 3., we have just shown that $AA = \{a^n : n \geq 0\}$. Since the regular expression a^* describes AA, this language is regular.