COMP 3803 — Fall 2025 — Problem Set 3

Question 1: Is the language

$$\{a^k b^\ell a^m : k \geq 0, \ell \geq 0, m \geq 0, k + \ell + m \geq 2025\}$$

regular? As always, justify your answer.

Question 2: For any string $w \in \{a, b\}^*$, we denote the number of a's in w by $N_a(w)$, and we denote the number of b's in w by $N_b(w)$. Consider the language

$$A = \{w \in \{a, b\}^* : N_a(w) = N_b(w)\}.$$

Assume that we are going to use the Pumping Lemma to prove that A is not regular. As always, we assume that A is regular. The Pumping Lemma gives us a pumping length p.

1. Explain in a few sentences why we may assume that p is even.
2. Given that p is even, can we choose the string $s = a^{p/2} b^{p/2}$ to obtain a contradiction?

Question 3: Use the Pumping Lemma to prove that the following languages are not regular.

1. $\{a^k b^\ell a^m : k \geq 0, \ell \geq 0, m \geq 0, \text{ and } k = \ell \text{ or } \ell \neq m\}$. The alphabet is $\{a, b\}$.
2. $\{a^m b^n : m \geq 0, n \geq 0, \ m + n \text{ is a prime number}\}$. The alphabet is $\{a, b\}$.
3. $\{a^m (ab)^n (ca)^{2m} : m \geq 1, n \geq 1\}$. The alphabet is $\{a, b, c\}$.
4. $\{a^i b^m c^n : i \geq 1, m \geq 1, n \geq m + 1\}$. The alphabet is $\{a, b, c\}$.

Question 4: Even though Justin Bieber is enjoying COMP 3803, he is confused about the Pumping Lemma. Here is Justin's confusion:

1. Justin knows that any finite language is regular. Can you prove this in a few sentences?
2. Let A be a finite regular language. By the Pumping Lemma, there is an integer pumping length $p \geq 1$. Take an arbitrary string s in A, with $|s| \geq p$. By the Pumping Lemma, we can write $s = xyz$, such that for every integer $i \geq 0$, the string xy^iz is in A. Since there are infinitely many integers $i \geq 0$, this seems to imply that A contains infinitely many strings. Therefore, the Pumping Lemma is not valid for finite languages.

   Is Justin's reasoning correct?

Question 5: Let A be an arbitrary regular language, let M be a DFA that accepts A, and let p be the number of states of M.

Prove that A is non-empty if and only if there is a string s in A whose length is strictly less than p.
Hint: There is a reason why the letter p is used to denote the number of states of M.

Question 6: Consider the context-free grammar $G = (V, \Sigma, R, S)$, where the set of variables is $V = \{S, A, B\}$, the set of terminals is $\Sigma = \{a, b\}$, the start variable is S, and the rules are as follows:

$$\begin{array}{lcl} S & \rightarrow & abB \\ A & \rightarrow & \varepsilon \mid aaBb \\ B & \rightarrow & bbAa \end{array}$$

Prove that the language $L(G)$ that is generated by G is equal to

$$L(G) = \{ab(bbaa)^n bba(ba)^n : n \geq 0\}.$$

(Remember: To prove that two sets X and Y are equal, you have to prove that $X \subseteq Y$ and $Y \subseteq X$.)

Question 7: Give context-free grammars that generate the following languages. For each case, justify your answer.
(7.1) $\{a^{n+3}b^n : n \geq 0\}$. The set of terminals is equal to $\{a, b\}$.
(7.2) $\{a^n b^m : n \geq 0, m \geq 0, 2n \leq m \leq 3n\}$. The set of terminals is equal to $\{a, b\}$.
(7.3) $\{a^m b^n c^n : m \geq 0, n \geq 0\}$. The set of terminals is equal to $\{a, b, c\}$.