Theorem: The relation $\preceq$ is transitive:

$$L \preceq L' \quad \land \quad L' \preceq L'' \quad \Rightarrow \quad L \preceq L''.$$  

Proof:

\[ \begin{array}{c}
\text{Input } x \\
\text{for } L \\
\xrightarrow{f} \\
\text{Input } y = f(x) \text{ for } L' \\
\xrightarrow{g} \\
\text{Input } g(y) \text{ for } L''
\end{array} \]

$x \in L \quad \Rightarrow \quad y = f(x) \in L' \quad \Rightarrow \quad g(y) \in L''$

$\therefore \quad x \in L \quad \Rightarrow \quad g(f(x)) \in L''.$

Reduction from $L$ to $L''$ is given by the function $g \circ f$.

Given $x$, $(g \circ f)(x) = g(f(x))$ can be computed in time that is polynomial in the length of $x$. (Why?)
Definition:

1. Language $L$ is **NP-hard** if
   for every $L'$ in NP: $L' \leq_{p} L$.

2. Language $L$ is **NP-complete** if
   - $L \in \text{NP}$ and
   - for every $L'$ in NP: $L' \leq_{p} L$.

In English: $L$ is NP-complete means
   - $L$ is in NP,
   - $L$ is at least as difficult as every problem in NP.

$\therefore L$ belongs to the most difficult problems in NP.

[This is what we wanted on page 182]
Theorem: Assume $L$ is NP-complete. Then:

$$L \in \mathbb{P} \iff \mathbb{P} = \mathbb{NP}.$$ 

Proof:

In formally:

-if $L \in \mathbb{P}$: $L$ is easy,

$L$ is NP-complete: $L$ belongs to the most difficult problems in NP.

\[ \therefore \text{ the most difficult problem in } \mathbb{NP} \text{ is easy} \]
\[ \therefore \text{ all problems in } \mathbb{NP} \text{ are easy} \]
\[ \therefore \mathbb{P} = \mathbb{NP}. \]

formally:

$\iff$ Assume $\mathbb{P} = \mathbb{NP}$.

Since $L$ is NP-complete: $L \in \mathbb{NP}$.

$\therefore L \in \mathbb{P}$.

Assume $L \in \mathbb{P}$.

We have to show that $\mathbb{P} = \mathbb{NP}$.

We know that $\mathbb{P} \subseteq \mathbb{NP}$. 
To show that \( NP \subset P \):

Let \( L' \in NP \).

Since \( L \) is \( NP \)-complete: \( L' \leq_p L \).

Since \( L \in P \): \( L' \in P \) (see page 184).

\[ \square \]

**Theorem:**

\[
\begin{align*}
L & \text{ NP-complete} \\
L & \leq_p L' \\
L' & \in NP
\end{align*}
\]

\[
\Rightarrow L' \text{ NP-complete.}
\]

**Proof:**

'Informally:

\( L \) is at least as difficult as every problem in \( NP \)

and

\( L' \) at least as difficult as \( L \)

\( \Rightarrow \) \( L' \) is at least as difficult as every problem in \( NP \).
formally: To show that \( L' \) is NP-complete, we have to show:

* \( L' \in \text{NP} \): this is given.
* for each \( L'' \in \text{NP} \): \( L'' \leq_{p} L' \).

why is this true:

* since \( L \) is NP-complete: \( L'' \leq_{p} L \).
* we are given: \( L \leq_{p} L' \).
* by transitivity (page 203): \( L'' \leq_{p} L' \).

How to use this: To show that \( L' \) is NP-complete:

1. show that \( L' \in \text{NP} \).
2. look for a problem \( L \) that is "similar" to \( L' \) and that is known to be NP-complete.
3. show that \( L \leq_{p} L' \).
In order to apply this, we need a first

NP-complete problem:

We need one language $L$ in NP such that

$\text{HAMCYCLE} \leq^p L$,

$\text{TSP} \leq^p L$,

$\text{SUBSETSUM} \leq^p L$,

$\text{CLIQUE} \leq^p L$,

$\text{INDEPENDENT-SET} \leq^p L$,

$\text{VERTEX-COVER} \leq^p L$,

$\text{3SAT} \leq^p L$,

$\text{3COLOR} \leq^p L$,

$\vdots$

for every $L'$ in NP: $L' \leq^p L$.

Not obvious that NP-complete problems exist!
1971: Stephen Cook proved that SAT is NP-complete.

*Independentely in Russia:*

1972: Leonid Levin proved that a certain tiling problem is NP-complete.