Midterm COMP 3804

February 17, 2011

Student Name:
Student Number:
25 marks total

Question 1: (9 marks) The Hadamard matrices $H_1, H_2, H_3, \ldots$ are recursively defined as follows:

$$H_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

and, for $k \geq 2$,

$$H_k = \begin{pmatrix} H_{k-1} & H_{k-1} \\ H_{k-1} & -H_{k-1} \end{pmatrix}.$$

Observe that, for $k \geq 1$, $H_k$ has $2^k$ rows and $2^k$ columns.

If $x$ is a column vector of length $2^k$, then $H_kx$ is the column vector of length $2^k$ obtained by multiplying the matrix $H_k$ with the vector $x$.

Describe, in plain English, a divide-and-conquer algorithm $\text{MULT}(k, x)$, that does the following:

**Input:** A positive integer $k$ and a column vector $x$ of length $n = 2^k$.

**Output:** The column vector $H_kx$ (having length $n$).

The running time of your algorithm must be $O(n \log n)$. Justify your answer. (If you encounter a recurrence relation that we have seen in class, then you can give its solution by writing “as we have seen in class, the solution to this recurrence is $O(\text{BLAHBLAH})$”.)
Question 2: (7 marks) Let $A[1\ldots n]$ be a min-heap storing $n$ distinct numbers, and let $z$ be the third smallest number among them. Give all possible indices $i$ for which $z$ can be equal to $A[i]$. (Equivalently, give all indices $j$ such that $z$ cannot be stored at $A[j]$.) Justify your answer.
**Question 3: (9 marks)** Let $G = (V, E)$ be a directed acyclic graph, in which each vertex $v$ has a price $p(v)$. For every vertex $u$ in $V$, define

$$cost(u) = \text{price of the cheapest vertex that is reachable from } u \text{ (including } u \text{ itself)}.$$ 

For example, in the graph below, where the price is shown for each vertex, the $cost$-values of the vertices $A, B, C, D, E, F$ are 2, 1, 4, 1, 4, 5, respectively.

Describe, in plain English, an algorithm that computes, in $O(|V| + |E|)$ time, the value $cost(u)$ for every vertex $u$. Justify your answer.

*Hint:* You may use any algorithm that was presented in class. Handle the vertices $u$ in a particular order.