Midterm COMP 3804

February 17, 2011

Student Name:

Student Number:

25 marks total

Question 1: (9 marks) The Hadamard matrices $H_1, H_2, H_3, \ldots$ are recursively defined as follows:

$$H_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

and, for $k \geq 2$,

$$H_k = \left(\begin{array}{c|c} H_{k-1} & H_{k-1} \\ \hline H_{k-1} & -H_{k-1} \end{array} \right).$$

Observe that, for $k \geq 1$, H_k has 2^k rows and 2^k columns.

If x is a column vector of length 2^k, then $H_k x$ is the column vector of length 2^k obtained by multiplying the matrix H_k with the vector x.

Describe, in plain English, a divide-and-conquer algorithm $\textsc{Mult}(k, x)$, that does the following:

Input: A positive integer k and a column vector x of length $n = 2^k$.

Output: The column vector $H_k x$ (having length n).

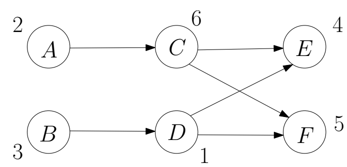
The running time of your algorithm must be $O(n \log n)$. Justify your answer. (If you encounter a recurrence relation that we have seen in class, then you can give its solution by writing "as we have seen in class, the solution to this recurrence is $O(BLAHBLAH)$".)

Question 2: (7 marks) Let $A[1 \dots n]$ be a min-heap storing n distinct numbers, and let z be the third smallest number among them. Give all possible indices i for which z can be equal to $A[i]$. (Equivalently, give all indices j such that z cannot be stored at $A[j]$.) Justify your answer.

Question 3: (**9 marks**) Let $G = (V, E)$ be a directed acyclic graph, in which each vertex v has a price $p(v)$. For every vertex u in V, define

$$cost(u) = \text{ price of the cheapest vertex that is reachable from } u \text{ (including } u \text{ itself).}$$

For example, in the graph below, where the price is shown for each vertex, the *cost*-values of the vertices A, B, C, D, E, F are $2, 1, 4, 1, 4, 5$, respectively.



Describe, in plain English, an algorithm that computes, in $O(|V| + |E|)$ time, the value $cost(u)$ for every vertex u. Justify your answer.

Hint: You may use any algorithm that was presented in class. Handle the vertices u in a particular order.